Lecture 5b

Legendre functions

Legendre's equation

In the previous lecture we obtained the following equation for the θ dependence of a solution to the Helmholtz equation in spherical coordinates

$$g''+\frac{\cos\theta}{\sin\theta}g'+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]g=0 \tag{1}$$

This is one form of the *associated Legendre equation*. The trig functions are not convenient to work with directly. Let's make the substitutions

$$\begin{gathered} x=\cos\theta \\ y(x)=g(\theta) \end{gathered} \tag{2}$$

and note that $0\le\theta\le\pi$ corresponds to $1\ge x\ge -1$. We have

$$\frac{dg}{d\theta}=\frac{dy}{dx}\frac{dx}{d\theta}=-\sin\theta\frac{dy}{dx}=-\sqrt{1-x^2}\frac{dy}{dx} \tag{3}$$

and

$$\begin{aligned} \frac{d^2g}{d\theta^2}&=-\frac{d}{d\theta}\sin\theta\frac{dy}{dx} \\ &=\sin^2\theta\frac{d^2y}{dx^2}-\cos\theta\frac{dy}{dx} \\ &=(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx} \end{aligned} \tag{4}$$

The associated Legendre equation becomes

$$(1-x^2)y''-2xy'+\left[n(n+1)-\frac{m^2}{1-x^2}\right]y=0 \tag{5}$$

This form is much more convenient to work with. The special case $m=0$ gives the *ordinary Legendre equation*

$$(1-x^2)y''-2xy'+n(n+1)y=0 \tag{6}$$

From the previous lecture we know that $m=0$ results in fields that have no ϕ dependence. We will solve the ordinary Legendre equation first and then use that solution to solve the associated equation.

Legendre functions

The standard form of the ordinary Legendre equation is

$$y''-\frac{2x}{1-x^2}y'+\frac{\nu(\nu+1)}{1-x^2}y=0 \tag{7}$$

were we've used ν instead of n as our constant to emphasize its complete generality (although it will typically be an integer in our applications). The coefficient functions are well-behaved at $x=0$ so we can find a solution of the form

$$y=\sum_{k=0}^{\infty}a_k x^k \tag{8}$$

However, note that the coefficient functions have singularities at $x=\pm1$ so we might expect problems at those points. More about that later.

Substituting the power series into the Legendre equation we get

$$\begin{gathered} (1-x^2)\sum_{k=2}^{\infty}k(k-1)a_k x^{k-2}-2x\sum_{k=1}^{\infty}k a_k x^{k-1} \\ +\nu(\nu+1)\sum_{k=0}^{\infty}a_k x^k=0 \end{gathered} \tag{9}$$

Rearranging we obtain

$$\sum_{k=2}^{\infty}k(k-1)a_k x^{k-2}=\sum_{k=0}^{\infty}\left[k(k-1)+2k-\nu(\nu+1)\right]a_k x^k \tag{10}$$

which can be written

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k=\sum_{k=0}^{\infty}\left[k(k-1)+2k-\nu(\nu+1)\right]a_k x^k \tag{11}$$

This gives us the recursion formula for the coefficients

$$a_{k+2}=\frac{(k+1+\nu)(k-\nu)}{(k+2)(k+1)}a_k \tag{12}$$

where we've replaced $k(k-1)+2k-\nu(\nu+1)$ by its equivalent $(k+1+\nu)(k-\nu)$. Clearly we can form two solutions, one starting with a_0 and containing even powers of x and one starting with a_1 and containing odd powers of x. Taking $a_0=a_1=1$ we can write

$$\begin{aligned} w_0(x)=1&+\frac{(1+\nu)(-\nu)}{2!}x^2 \\ &+\frac{(1+\nu)(-\nu)(3+\nu)(2-\nu)}{4!}x^4+\cdots \end{aligned} \tag{13}$$

and

$$\begin{aligned} w_1(x)=x&+\frac{(2+\nu)(1-\nu)}{3!}x^3 \\ &+\frac{(2+\nu)(1-\nu)(4+\nu)(3-\nu)}{5!}x^5+\cdots \end{aligned} \tag{14}$$

Let's use the ratio test to investigate the convergence of these series. We have

$$\left|\frac{a_{k+2}x^{k+2}}{a_k x^k}\right|=\frac{(k+1+\nu)(k-\nu)}{(k+2)(k+1)}x^2\to x^2 \tag{15}$$

as $k\to\infty$. The series converges absolutely if this limit is bounded by a number less than unity. Clearly this will only be the case for $|x|<1$. Therefore, we expect that these series will have convergence problems at $x=\pm1$. In other words, the functions w_0, w_1 likely have singularities at $x=\pm1$. Since $x=\cos\theta$ this corresponds to singularities at $\theta=0,\pi$, that is, at the “poles” of the unit sphere.

However, from (12) we see that if $\nu=n$ is a non-negative integer then one of these series will terminate at the x^n term. Therefore one solution will be an n[th] order polynomial and have no singularities. We call these solutions the *Legendre polynomials*. If $\nu=n$ is even, then $w_0(x)$ will be a polynomial while $w_1(x)$ will an infinite series with singularities at $x=\pm 1$. If $\nu=n$ is odd, then $w_1(x)$ will be a polynomial while $w_0(x)$ will have the singularities.

It is convenient to choose as our two basis function some linear combinations of $w_0(x), w_1(x)$, which we denote by $P_\nu(x), Q_\nu(x)$ as

$$\begin{aligned} P_\nu(x) &= a_0 w_0(x) + a_1 w_1(x) \\ Q_\nu(x) &= b_0 w_0(x) + b_1 w_2(x) \end{aligned} \tag{16}$$

such that for $\nu=n$ an integer (even or odd) $P_n(x)$ is always the polynomial. We can do this by defining the *Legendre function of the first kind of degree* ν as

$$P_\nu(x) = \left(-\frac{1}{2}\right)! \left[\frac{w_0(x)}{\left(\frac{\nu}{2}\right)!\left(-\frac{\nu+1}{2}\right)!} + \frac{\nu\, w_1(x)}{\left(-\frac{\nu}{2}\right)!\left(\frac{\nu-1}{2}\right)!} \right] \tag{17}$$

To see how this works, recall that the factorial of a negative integer is infinite. If $\nu=n$ is a positive, even integer, then $(-\nu/2)!$ is infinite (it's the factorial of a negative integer), the second term in (17) is zero and

$$P_\nu(x) = \frac{\left(-\frac{1}{2}\right)!}{\left(\frac{n}{2}\right)!\left(-\frac{n+1}{2}\right)!} w_0(x) \tag{18}$$

is a finite constant times the polynomial $w_0(x)$. Likewise, if $\nu=n$ is a positive, odd integer, then $(-[\nu+1]/2)!$ is infinite and

$$P_\nu(x) = \frac{\nu\left(-\frac{1}{2}\right)!}{\left(-\frac{\nu}{2}\right)!\left(\frac{\nu-1}{2}\right)!} w_1(x) \tag{19}$$

is a finite constant times the polynomial $w_1(x)$. The function $Q_\nu(x)$ is the *Legendre function of the second kind of degree* ν. It never reduces to a polynomial and always exhibits singularities at $x=\pm 1$.

Through a subtle application of complex contour integration techniques one can derive the integral representation

$$P_\nu(\cos\theta) = \frac{\sqrt{2}}{\pi} \int_0^\theta \frac{\cos([\nu+1/2]t)}{\sqrt{\cos(t)-\cos(\theta)}}\, dt \tag{20}$$

valid for all ν (see Lebedev for details).

Integer-order Legendre functions

As mentioned previous, for $\nu=n$ the Legendre functions of the first kind reduce to polynomials. These are conveniently represented by the Rodriguez formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{d x^n} (x^2-1)^n \tag{21}$$

The corresponding Legendre function of the second kind is

$$Q_n(x) = \frac{1}{2} P_n(x) \ln\left(\frac{1+x}{1-x}\right) - W_{n-1}(x) \tag{22}$$

where $W_{n-1}(x)$ is a polynomial of order $n-1$. The 0[th] order functions are

$$\begin{aligned} P_0(x) &= 1 \\ Q_0(x) &= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \end{aligned} \tag{23}$$

while the 1[st] order functions are

$$\begin{aligned} P_1(x) &= x \\ Q_1(x) &= \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) - 1 \end{aligned} \tag{24}$$

Higher order function can be derived from the recursion formula

$$R_{n+1}(x) = \frac{2n+1}{n+1}\, x\, R_n(x) - \frac{n}{n+1} R_{n-1}(x) \tag{25}$$

Associated Legendre functions

We have solved the "ordinary" Legendre equation and found the two solutions $P_\nu(x), Q_\nu(x)$. However, our final goal is to solve the associated Legendre equation. This is

$$(1-x^2)\, y'' - 2x\, y' + \left[\nu(\nu+1) - \frac{m^2}{1-x^2}\right] y = 0 \tag{26}$$

The substitution $y(x) = (1-x^2)^{m/2} u(x)$ produced the following equation for u

$$(1-x^2)\, u'' - 2(m+1)\, x\, u' + (\nu-m)(\nu+m+1)\, u = 0 \tag{27}$$

Given that $P_\nu(x), Q_\nu(x)$ solve

$$(1-x^2)\, y'' - 2x\, y' + \nu(\nu+1)\, y = 0 \tag{28}$$

by direct substitution one can verify that for integer *m*, (27) is solved by the functions

$$u = \begin{cases} \dfrac{d^m}{d x^m} P_\nu(x) \\ \dfrac{d^m}{d x^m} Q_\nu(x) \end{cases} \tag{29}$$

We therefore have the *associated Legendre functions of the first and second kind*

$$\begin{aligned} P_\nu^m(x) &= (-1)^m (1-x^2)^{m/2} \frac{d^m}{d x^m} P_\nu(x) \\ Q_\nu^m(x) &= (-1)^m (1-x^2)^{m/2} \frac{d^m}{d x^m} Q_\nu(x) \end{aligned} \tag{30}$$

of order ν *and degree m*. In the Appendix we derive several of these for integer orders and degrees. Notice that in this case both functions have closed-form expressions. Further note that since $P_n(x)$ is an n^th degree polynomial, $P_n^m(x) \equiv 0$ for $m > n$.

General solution of Helmholtz equation

A general solution of the Helmholtz equation that is periodic in ϕ is

$$A_r = r \begin{Bmatrix} j_\nu(\beta r) \\ y_\nu(\beta_r) \end{Bmatrix} \begin{Bmatrix} P_\nu^m(\cos\theta) \\ Q_\nu^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix} \tag{31}$$

where ν is arbitrary. If we require a solution that is finite for $0 \le \theta \le \pi$ then we must exclude the 2^nd kind of Legendre functions and we must have $\nu = n$ an integer

$$A_r = r \begin{Bmatrix} j_n(\beta r) \\ y_n(\beta_r) \end{Bmatrix} P_n^m(\cos\theta) \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix} \tag{32}$$

Finally, if we require the solution to be finite at the origin we are left with

$$A_r = r\, j_n(\beta r)\, P_n^m(\cos\theta) \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix} \tag{33}$$

for $n = 0, 1, 2, \ldots$ and $0 \le m \le n$.

References

1. Hobson, E. W., *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, 1931.
2. Lebedev, N. N. *Special Functions and Their Applications.* Dover Publications 1972. ISBN 0-486-60624-4.
3. Carrier, G. F., M. Krook and C. E. Pearson. *Functions of a Complex Variable.* Hod Books 1983. ISBN 07-010089-6.

Appendix

The following Maxima code generates the functions $P_n^m(x)$.

```
N:3$

P[0,0]:1$
for n:1 step 1 thru N do (
   P[n,0]:expand(
      diff((x^2-1)^n,x,n)/((2^n)*n!)),
   for m:1 step 1 thru n do (
      P[n,m]:factor(((-1)^m)
         *((1-x^2)^(m/2))*diff(P[n,0],x,m))
   )
)$

for n:0 step 1 thru N do (
   for m:0 step 1 thru n do (
      display(P[n,m])
   )

)$
```

The output is (notation: $P_{n,m}=P_n^m(x)$)

$$P_{0,0}=1$$

$$P_{1,0}=x$$

$$P_{1,1}=-\sqrt{1-x^2}$$

$$P_{2,0}=\frac{3\,x^2}{2}-\frac{1}{2}$$

$$P_{2,1}=-3\,x\,\sqrt{1-x^2}$$

$$P_{2,2}=-3\,(x-1)\,(x+1)$$

$$P_{3,0}=\frac{5\,x^3}{2}-\frac{3\,x}{2}$$

$$P_{3,1}=-\frac{3\,\sqrt{1-x^2}\,(5\,x^2-1)}{2}$$

$$P_{3,2}=-15\,(x-1)\,x\,(x+1)$$

$$P_{3,3}=15\,(x-1)\,(x+1)\,\sqrt{1-x^2}$$

The following Maxima code generates the functions $Q_n^m(x)$.

```
N:3$

a[0]:1$ b[0]:0$
a[1]:x$ b[1]:-1$
Q[0,0]:(1/2)*log((1+x)/(1-x))$
for n:1 step 1 thru N do (
   Q[n,0]:expand(a[n])*(log((1+x)/
      (1-x))/2)+expand(b[n]),
   a[n+1]:ratsimp(((2*n+1)/(n+1))*x
      *a[n]-(n/(n+1))*a[n-1]),
   b[n+1]:ratsimp(((2*n+1)/(n+1))*x
      *b[n]-(n/(n+1))*b[n-1]),
   for m:1 step 1 thru n do (
      logfact:ratsimp(diff(a[n],x,m)),
      ratterm:ratsimp(diff(Q[n,0],x,m)
         -logfact*(log((1+x)/(1-x))/2)),
      Q[n,m]:factor((-1)^m
         *(1-x^2)^(m/2)*logfact)
         *log((1+x)/(1-x))/2,
      Q[n,m]:Q[n,m]+factor((-1)^m
         *(1-x^2)^(m/2)*ratterm)
   )
)$

for n:0 step 1 thru N do (
   for m:0 step 1 thru n do (
      display(Q[n,m])
   )
)$
```

The output is (notation: $Q_{n,m}=Q_n^m(x)$)

$$Q_{0,0}=\frac{\log\left(\frac{x+1}{1-x}\right)}{2}$$

$$Q_{1,0}=\frac{x\log\left(\frac{x+1}{1-x}\right)}{2}-1$$

$$Q_{1,1}=\frac{x\,\sqrt{1-x^2}}{(x-1)\,(x+1)}-\frac{\sqrt{1-x^2}\log\left(\frac{x+1}{1-x}\right)}{2}$$

$$Q_{2,0}=\frac{\left(\frac{3x^2}{2}-\frac{1}{2}\right)\log\left(\frac{x+1}{1-x}\right)}{2}-\frac{3\,x}{2}$$

$$Q_{2,1}=\frac{\sqrt{1-x^2}\,(3\,x^2-2)}{(x-1)\,(x+1)}-\frac{3\,x\,\sqrt{1-x^2}\log\left(\frac{x+1}{1-x}\right)}{2}$$

$$Q_{2,2}=\frac{x\,(3\,x^2-5)}{(x-1)\,(x+1)}-\frac{3\,(x-1)\,(x+1)\log\left(\frac{x+1}{1-x}\right)}{2}$$

$$Q_{3,0}=\frac{\left(\frac{5x^3}{2}-\frac{3x}{2}\right)\log\left(\frac{x+1}{1-x}\right)}{2}-\frac{5\,x^2}{2}+\frac{2}{3}$$

$$Q_{3,1}=\frac{x\,\sqrt{1-x^2}\,(15\,x^2-13)}{2\,(x-1)\,(x+1)}-\frac{3\,\sqrt{1-x^2}\,(5\,x^2-1)\log\left(\frac{x+1}{1-x}\right)}{4}$$

$$Q_{3,2}=\frac{15\,x^4-25\,x^2+8}{(x-1)\,(x+1)}-\frac{15\,(x-1)\,x\,(x+1)\log\left(\frac{x+1}{1-x}\right)}{2}$$

$$Q_{3,3}=\frac{15\,(x-1)\,(x+1)\,\sqrt{1-x^2}\log\left(\frac{x+1}{1-x}\right)}{2}-\frac{x\,\sqrt{1-x^2}\,(15\,x^4-40\,x^2+33)}{(x-1)^2\,(x+1)^2}$$