Lecture 5a

Vectors and operators in spherical coordinates

Unit vectors and metric coefficients

Spherical coordinates r, θ, ϕ are defined by

$$x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \tag{1}$$

They are the coordinates of choice in problems with spherical boundaries. Since *z* is no longer one of the coordinates we will not be able to use A_z and F_z to specify the fields. This will lead to some subtleties. Let's first derive the unit vectors and the metric coefficients. Since

$$\frac{\partial \mathbf{r}}{\partial r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2}$$

we have

$$h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = \sqrt{\sin^2\theta[\cos^2\phi + \sin^2\phi] + \cos^2\theta} = 1 \tag{3}$$

and

$$\hat{a}_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{4}$$

Likewise

$$\frac{\partial \mathbf{r}}{\partial \theta} = r(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \tag{5}$$

so that

$$h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r\sqrt{\cos^2\theta[\cos^2\phi + \sin^2\phi] + \sin^2\theta} = r \tag{6}$$

and

$$\hat{a}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \tag{7}$$

Finally

$$\frac{\partial \mathbf{r}}{\partial \phi} = r(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0) \tag{8}$$

so

$$h_\phi = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = r\sqrt{\sin^2\theta[\cos^2\phi + \sin^2\phi]} = r\sin\theta \tag{9}$$

and

$$\hat{a}_\phi = (-\sin\phi, \cos\phi, 0) \tag{10}$$

Using the metric coefficients, we are now ready to derive the various differential operators.

Differential operators

Using the results of Lecture 1b for a general orthogonal coordinate system, and identifying $u = r, v = \theta, w = \phi$ we have the gradient

$$\nabla f = \hat{a}_r \frac{\partial f}{\partial r} + \hat{a}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{a}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \tag{11}$$

The divergence is

$$\begin{aligned} \nabla\cdot\mathbf{A} = & \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}(r^2\sin\theta A_r) \\ & + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(r\sin\theta A_\theta) \\ & + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \phi}(r A_\phi) \end{aligned} \tag{12}$$

Simplifying we obtain

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{\partial}{\partial \phi}A_\phi\right] \tag{13}$$

The curl is

$$\begin{aligned} \nabla\times\mathbf{A} = & \frac{\hat{a}_r}{r^2\sin\theta}\left[\frac{\partial}{\partial \theta}(r\sin\theta A_\phi) - \frac{\partial}{\partial \phi}(r A_\theta)\right] \\ & + \frac{\hat{a}_\theta}{r\sin\theta}\left[\frac{\partial}{\partial \phi}A_r - \frac{\partial}{\partial r}(r\sin\theta A_\phi)\right] \\ & + \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial}{\partial \theta}A_r\right] \end{aligned} \tag{14}$$

Simplified (somewhat) we get

$$\begin{aligned} \nabla\times\mathbf{A} = & \frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial}{\partial \phi}A_\theta\right] \\ & + \frac{\hat{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}A_r - \frac{\partial}{\partial r}(r A_\phi)\right] \\ & + \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial}{\partial \theta}A_r\right] \end{aligned} \tag{15}$$

The Laplacian is

$$\begin{aligned} \nabla^2 f = & \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) \\ & + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\frac{r\sin\theta}{r}\frac{\partial f}{\partial \theta}\right) \\ & + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{r}{r\sin\theta}\frac{\partial f}{\partial \phi}\right) \end{aligned} \tag{16}$$

This simplifies to

$$\begin{aligned} \nabla^2 f = & \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) \\ & + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \end{aligned} \tag{17}$$

The Laplacian of a vector field is by definition

$$\nabla^2 \mathbf{A} \equiv \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A} \quad (18)$$

In cylindrical coordinates we avoided this expression because for $\mathbf{A}=\hat{a}_z A_z$ will simply had $\nabla^2\mathbf{A}=\hat{a}_z\nabla^2 A_z$. Things are not as simple in spherical coordinates.

Helmholtz equation

The Helmholtz equation, which is independent of any coordinate system, is

$$\nabla^2\mathbf{A}+\beta^2\mathbf{A}=0 \quad (19)$$

In general this will read

$$\nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}+\beta^2\mathbf{A}=0 \quad (20)$$

This is a pretty ugly vector equation. However, recall from Lecture 2c that

$$\nabla\cdot\mathbf{A}=\frac{j}{\omega}\left(\nabla\cdot\mathbf{E}+\nabla^2\psi\right) \quad (21)$$

where ψ is an arbitrary scalar field. The divergence of **A** is completely unconstrained by physics. We can make it be anything we want. Let's see if we can use this freedom to reduce our vector equation to a single scalar equation. Let's take

$$\mathbf{A}=\hat{a}_r A_r \quad (22)$$

From the curl expression we see that these will be TM^r modes in which $H_r\equiv 0$. We have

$$\nabla\times\mathbf{A}=\frac{\hat{a}_\theta}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}A_r-\frac{\hat{a}_\phi}{r}\frac{\partial}{\partial\theta}A_r \quad (23)$$

Applying another curl we obtain

$$\begin{aligned}\nabla\times\nabla\times\mathbf{A} &\\ =\frac{\hat{a}_r}{r\sin\theta}&\left[-\frac{\partial}{\partial\theta}\left(\sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}A_r\right)-\left(\frac{\partial}{\partial\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_r\right)\right] \\ &+\frac{\hat{a}_\theta}{r}\frac{\partial}{\partial r}\left(r\frac{1}{r}\frac{\partial}{\partial\theta}A_r\right)+\frac{\hat{a}_\phi}{r}\frac{\partial}{\partial r}\left(r\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_r\right)\end{aligned} \quad (24)$$

Therefore

$$\begin{aligned}-\nabla\times\nabla\times\mathbf{A} &\\ =\frac{\hat{a}_r}{r^2\sin\theta}&\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}A_r\right)+\frac{1}{\sin\theta}\left(\frac{\partial^2}{\partial^2\phi^2}A_r\right)\right] \\ &-\frac{\hat{a}_\theta}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}A_r-\frac{\hat{a}_\phi}{r\sin\theta}\frac{\partial}{\partial r}\frac{\partial}{\partial\phi}A_r\end{aligned} \quad (25)$$

Since $\beta^2\mathbf{A}$ has only an *r* component, let's see if we can use $\nabla\cdot\mathbf{A}$ to cancel the θ,ϕ components of $-\nabla\times\nabla\times\mathbf{A}$. Let's call $\nabla\cdot\mathbf{A}=f$ and remember that this scalar function can be anything we want it to be. $\nabla(\nabla\cdot\mathbf{A})$ is

$$\nabla f=\hat{a}_r\frac{\partial f}{\partial r}+\hat{a}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta}+\hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi} \quad (26)$$

If

$$\frac{1}{r}\frac{\partial f}{\partial\theta}-\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}A_r=0 \quad (27)$$

and

$$\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}-\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\frac{\partial}{\partial\phi}A_r=0 \quad (28)$$

then we will have achieved our goal. This is true if we take

$$f=\frac{\partial}{\partial r}A_r \quad (29)$$

The operator $\nabla(\nabla\cdot\mathbf{A})-\nabla\times\nabla\times\mathbf{A}$ now has only the *r* component

$$\frac{\partial^2 A_r}{\partial r^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2} \quad (30)$$

where $\partial^2 A_r/\partial r^2$ is the *r* component of ∇f. We could add $\beta^2 A_r$ to this and apply separation of variables. However, from a mathematical point of view it would be much preferred to if we could employ the scalar Laplacian operator. Now $\nabla^2 A_r$ is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_r}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2} \quad (31)$$

This is almost the same as (30), but the first terms differ. In fact

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_r}{\partial r}\right)=\frac{\partial^2 A_r}{\partial r^2}+\frac{2}{r}\frac{\partial A_r}{\partial r} \quad (32)$$

and we have an extra $(2/r)\partial A_r/\partial r$ term. However, note that

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{A_r}{r}\right)\right)=\frac{1}{r}\frac{\partial^2 A_r}{\partial r^2} \quad (33)$$

Therefore

$$\nabla^2\left(\frac{A_r}{r}\right)=\frac{1}{r}\left[\frac{\partial^2 A_r}{\partial r^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2}\right] \quad (34)$$

This tells us that the function $\psi=A_r/r$ satisfies the scalar Helmholtz equation

$$\nabla^2\psi+\beta^2\psi=0 \quad (35)$$

in spherical coordinates. If we solve this equation, then the vector potential is obtained from $A_r=r\psi$.

Separation of variables

Let's take

$$\psi = f(r) g(\theta) h(\phi) \tag{36}$$

The Helmholtz equation then reads

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f' g h\right)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta f g' h\right) + \frac{1}{r^2\sin^2\theta} f g h'' + \beta^2 f g h = 0 \tag{37}$$

Dividing through by *fgh* we obtain

$$\frac{1}{r^2 f}\frac{\partial}{\partial r}\left(r^2 f'\right)+\frac{1}{r^2 g\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta g'\right) + \frac{1}{r^2\sin^2\theta}\frac{h''}{h}+\beta^2=0 \tag{38}$$

Multiplying through by $r^2\sin^2\theta$ we arrive at

$$\frac{\sin^2\theta}{f}\frac{\partial}{\partial r}\left(r^2 f'\right)+\frac{\sin\theta}{g}\frac{\partial}{\partial\theta}\left(\sin\theta g'\right) + \frac{h''}{h}+\beta^2 r^2\sin^2\theta=0 \tag{39}$$

The h''/h term depends only on ϕ and no other term depends on ϕ. By our usual argument this term must be a constant. Let's write

$$\frac{h''}{h}=-m^2 \tag{40}$$

where m^2 is an arbitrary complex number (but *m* will typically be an integer) and

$$h=\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \tag{41}$$

Replacing h''/h in the Helmholtz equation and dividing through by $\sin^2\theta$ we arrive at

$$\frac{1}{f}\frac{\partial}{\partial r}\left(r^2 f'\right)+\beta^2 r^2+\frac{1}{g\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta g'\right)-\frac{m^2}{\sin^2\theta}=0 \tag{42}$$

Notice that all *r* dependence is in the first two terms and all θ dependence is in the last two terms. Therefore

$$\frac{1}{g\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta g'\right)-\frac{m^2}{\sin^2\theta}=-n(n+1) \tag{43}$$

where $-n(n+1)$ is an arbitrary complex constant (although *n* will usually be an integer). We have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta g'\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]g=0 \tag{44}$$

or

$$g''+\frac{\cos\theta}{\sin\theta}g'+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]g=0 \tag{45}$$

We will come back to this equation later. What remains of the Helmholtz equation is the *r* dependence

$$\frac{\partial}{\partial r}\left(r^2 f'\right)+\left[\beta^2 r^2-n(n+1)\right]f=0 \tag{46}$$

which expands to

$$r^2 f''+2 r f'+\left[\beta^2 r^2-n(n+1)\right]f=0 \tag{47}$$

Let's call

$$\begin{aligned}x&=\beta r\\ y(x)&=f(r)\end{aligned} \tag{48}$$

Using the Chain Rule we have

$$\frac{df}{dr}=\frac{dy}{dx}\frac{dx}{dr}=\beta y' \tag{49}$$

and our equation becomes

$$x^2 y''+2 x y'+\left[x^2-n(n+1)\right]y=0 \tag{50}$$

If it wasn't for the factor of 2 this would be in the form of Bessel's equation

$$x^2 y''+x y'+\left(x^2-\nu^2\right)y=0 \tag{51}$$

Since we've already solved Bessel's equation, it makes sense to look for a transformation that will covert our equation to Bessel's equation. Setting $y(x)=z(x)/\sqrt{x}$ we have

$$\begin{aligned}y'&=\frac{z'}{\sqrt{x}}-\frac{z}{2x^{3/2}}\\ y''&=\frac{z''}{\sqrt{x}}-\frac{z'}{x^{3/2}}+\frac{3}{4}\frac{z}{x^{5/2}}\end{aligned} \tag{52}$$

Substituting these expressions into (50) and multiplying through by $\sqrt{x}$ results in

$$x^2 z''+x z'+\left[x^2-(n+1/2)^2\right]z=0 \tag{53}$$

which is Bessel's equation with $\nu=n+1/2$. The solutions are linear combinations of the Bessel functions

$$z=\begin{Bmatrix}J_{n+1/2}(x)\\ Y_{n+1/2}(x)\end{Bmatrix} \tag{54}$$

Therefore $y(x)$ is $1/\sqrt{x}$ times a Bessel function. We are led to define the *spherical Bessel functions*

$$\begin{aligned}j_n(x)&=\sqrt{\frac{\pi}{2x}}J_{n+1/2}(x)\\ y_n(x)&=\sqrt{\frac{\pi}{2x}}Y_{n+1/2}(x)\end{aligned} \tag{55}$$

The *r* behavior of ψ is therefore a linear combination of $j_n(\beta r)$ and $y_n(\beta r)$. Since $A_r=r\psi$, our solutions will have the form

$$A_r = r\begin{Bmatrix} j_n(\beta r) \\ y_n(\beta_r) \end{Bmatrix}\begin{Bmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{Bmatrix}\begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix} \tag{56}$$

where $P_n^m(\cos\theta), Q_n^m(\cos\theta)$ are *Legendre functions*. We will derive these in the next lecture.

Fields

TM^r modes

Using (15) and $\mathbf{H} = (1/\mu)\nabla\times\mathbf{A}$, $\mathbf{E} = (-j/\omega\epsilon)\nabla\times\mathbf{H}$ with $\mathbf{A} = \hat{a}_r A_r$, we have

$$\boxed{\begin{aligned} H_r &= 0 \\ H_\theta &= \frac{1}{\mu r\sin\theta}\frac{\partial}{\partial\phi}A_r \\ H_\phi &= -\frac{1}{\mu r}\frac{\partial}{\partial\theta}A_r \\ E_r &= \frac{j}{\omega\mu\epsilon r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}A_r\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}A_r\right] \\ E_\theta &= -\frac{j}{\omega\mu\epsilon r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}A_r \\ E_\phi &= -\frac{j}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial r}\frac{\partial}{\partial\phi}A_r \end{aligned}} \tag{57}$$

TE^r modes

Using (15) and $\mathbf{E} = -(1/\epsilon)\nabla\times\mathbf{F}$, $\mathbf{H} = (j/\omega\mu)\nabla\times\mathbf{E}$ with $\mathbf{F} = \hat{a}_r F_r$, we have

$$\boxed{\begin{aligned} E_r &= 0 \\ E_\theta &= -\frac{1}{\epsilon r\sin\theta}\frac{\partial}{\partial\phi}F_r \\ E_\phi &= \frac{1}{\epsilon r}\frac{\partial}{\partial\theta}F_r \\ H_r &= \frac{j}{\omega\mu\epsilon r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}F_r\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}F_r\right] \\ H_\theta &= -\frac{j}{\omega\mu\epsilon r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}F_r \\ H_\phi &= -\frac{j}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial r}\frac{\partial}{\partial\phi}F_r \end{aligned}} \tag{58}$$

Appendix

The following Maxima code generates the spherical Bessel functions $j_n(x)$ and their Taylor series.

```
N:3$
j[0]:sin(x)/x$
j[1]:sin(x)/x^2-cos(x)/x$
for n:1 step 1 thru N-1 do (
  j[n+1]:ratsimp(((2*n+1)/x)*j[n]-j[n-1])
)$
for n:0 step 1 thru N do (
  display(j[n]),
  j[n]:taylor(j[n],x,0,n+2),
  display(j[n])
)$
```

The output is (notation $j_n = j_n(x)$)

$$j_0 = \frac{\sin x}{x}$$

$$j_0 = 1 - \frac{x^2}{6} + \cdots$$

$$j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_1 = \frac{x}{3} - \frac{x^3}{30} + \cdots$$

$$j_2 = -\frac{(x^2-3)\sin x + 3x\cos x}{x^3}$$

$$j_2 = \frac{x^2}{15} - \frac{x^4}{210} + \cdots$$

$$j_3 = -\frac{(6x^2-15)\sin x + (15x - x^3)\cos x}{x^4}$$

$$j_3 = \frac{x^3}{105} - \frac{x^5}{1890} + \cdots$$

The Taylor series show that $j_n(x)$ behaves as x^n for small x.