Lecture 4d

Wedge-shaped waveguides

Introduction

The Separation of Variables technique is useful in problems where each part of the boundary corresponds to a constant value of one of the coordinates. For example, if $A=f(\rho)g(\phi)h(z)$ then we can get $A=0$ on the surface $\rho=a$ by enforcing the scalar equation $f(a)=0$, or $\partial A/\partial\phi=0$ on $\phi=\phi_0$ by enforcing $g'(\phi_0)=0$ and so on. In the previous lecture we considered cylindrical waveguides with PEC surfaces at some value $\rho=a$. In this lecture we want to see what happens if we add surfaces at fixed values of ϕ. The result will be a wedge-shaped waveguide as illustrated below.

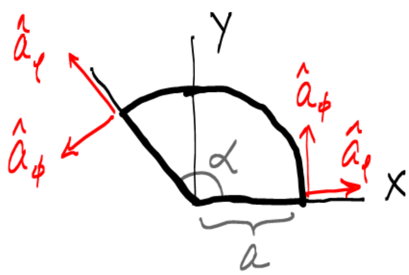

Figure 1: Cross section of wedge-shaped waveguide.

We will place one flat PEC surface at $\phi=0$ and another at $\phi=\alpha$. These surfaces will remove the "periodic in ϕ" requirement and impose additional boundary conditions which will modify our previous solutions.

TE^z modes

A general TE^z_{mn} mode which propagates in the *z* direction and is finite at the origin is described by

$$F_z(\rho,\phi,z)=J_\nu(\beta_\rho\rho)\begin{Bmatrix}\cos(\nu\phi)\\ \sin(\nu\phi)\end{Bmatrix}e^{-j\beta_z z} \tag{1}$$

where the brace notation represents an arbitrary linear combination of the corresponding functions, and ν need not be an integer. In order for the Helmholtz equation to be satisfied we require the condition

$$\beta_\rho^2+\beta_z^2=\omega^2\mu\epsilon \tag{2}$$

The boundary conditions are

$$\begin{gathered}\text{fields finite at } \rho=0\\ E_\phi=0 \text{ at } \rho=a\\ E_\rho=0 \text{ at } \phi=0,\alpha\end{gathered} \tag{3}$$

The first condition is met by using only the *J* Bessel function. Then, since

$$E_\rho=-\frac{1}{\epsilon\rho}\frac{\partial}{\partial\phi}F_z \tag{4}$$

the third condition requires

$$\frac{d}{d\phi}\begin{Bmatrix}\cos(\nu\phi)\\ \sin(\nu\phi)\end{Bmatrix}=\nu\begin{Bmatrix}-\sin(\nu\phi)\\ \cos(\nu\phi)\end{Bmatrix}=0 \tag{5}$$

at $\phi=0$ and at $\phi=\alpha$. The condition at $\phi=0$ tells us that we should use the $\cos(\nu\phi)$ factor for F_z since that will give a factor of $\sin(\nu\phi)$ in E_ρ. At $\phi=\alpha$ we then have

$$\sin(\nu\alpha)=0 \tag{6}$$

which requires

$$\nu=m\frac{\pi}{\alpha} \tag{7}$$

In general these will be non-integer values. However, if $\alpha=\pi/N$ for integer N then $\nu=mN$ is an integer. Our solution therefore have the form

$$F_z(\rho,\phi,z)=F_0J_\nu(\beta_\rho\rho)\cos(\nu\phi)e^{-j\beta_z z} \tag{8}$$

with ν given by (7). Note that $m=0$ is acceptable since $\cos 0=1$ does not vanish.

From the previous lecture we know that the condition at $\rho=a$ requires the derivative of the Bessel function to be zero, that is

$$J_\nu'(\beta_\rho a)=0 \tag{9}$$

Therefore

$$\beta_\rho=\frac{x'_{\nu n}}{a} \tag{10}$$

where

$$J_\nu'(x'_{\nu n})=0 \tag{11}$$

for $n=1,2,\ldots$. Therefore our modes are described by

$$\begin{gathered}F_z(\rho,\phi,z)=F_0J_\nu(\beta_\rho\rho)\cos(\nu\phi)e^{-j\beta_z z}\\ \nu=m\frac{\pi}{\alpha}\\ \beta_\rho=\frac{x'_{\nu n}}{a}\end{gathered} \tag{12}$$

for $m=0,1,2,\ldots$, $n=1,2,\ldots$. This is identical to the circular waveguide TE^z_{mn} modes except that *m* is replaced by ν and $\phi_0=0$. It follows that the **E** and **H** fields are given by equations (14) and (15) of Lecture 4c with $m\to\nu$. We have

$$E_\rho = \frac{\nu}{\epsilon\rho} F_0 J_\nu(\beta_\rho \rho)\sin(\nu) e^{-j\beta_z z}$$
$$E_\phi = \frac{\beta_\rho}{\epsilon} F_0 J_\nu{}'(\beta_\rho \rho)\cos(\nu\phi) e^{-j\beta_z z} \quad (13)$$
$$E_z = 0$$

and

$$H_\rho = -\frac{\beta_\rho \beta_z}{\omega\mu\epsilon} F_0 J_\nu{}'(\beta_\rho \rho)\cos(\nu\phi) e^{-j\beta_z z}$$
$$H_\phi = \frac{\nu\beta_z}{\omega\mu\epsilon\rho} F_0 J_\nu(\beta_\rho \rho)\sin(\nu\phi) e^{-j\beta_z z} \quad (14)$$
$$H_\rho = -j\frac{\beta_\rho^2}{\omega\mu\epsilon} F_0 J_\nu(\beta_\rho \rho)\cos(\nu\phi) e^{-j\beta_z z}$$

Resonant cavities

For rectangular waveguides we were able to form a resonate cavity by placing PEC surfaces at $z=0,c$. We can do the same with a circular or wedge waveguide. Let's consider the dominant TE_{11}^z mode and combine waves traveling the $\pm z$ directions.

$$F_z(\rho,\phi,z) = J_1\left(\beta_\rho \rho\right)\cos(\phi-\phi_0)\left(F_1 e^{-j\beta_z z} + F_2 e^{j\beta_z z}\right) \quad (15)$$

where $\beta_\rho = 1.841/a$. Let's put PEC planes at $z=0,c$. This adds the boundary conditions

$$E_\rho = E_\phi = 0 \ \text{ at } \ z=0,c \quad (16)$$

Since

$$E_\rho = -\frac{1}{\epsilon\rho}\frac{\partial}{\partial\phi} F_z$$
$$E_\phi = \frac{1}{\epsilon}\frac{\partial}{\partial\rho} F_z \quad (17)$$

and neither of these expressions involves a *z* derivative, we need to set

$$0 = F_1 + F_2$$
$$0 = F_1 e^{-j\beta_z c} + F_2 e^{j\beta_z c} \quad (18)$$

This requires that the *z* dependence of F_z be through a factor $\sin(\beta_z z)$ with $\beta_z = p\pi/c$ and $p=1,2,\ldots$. We then have

$$F_z(\rho,\phi,z) = F_0 J_1\left(\beta_\rho \rho\right)\cos(\phi-\phi_0)\sin(\beta_z z) \quad (19)$$

Since

$$\beta_\rho^2 + \beta_z^2 = \omega^2 \mu\epsilon \quad (20)$$

the frequency of the first of these resonant modes ($p=1$) is fixed by

$$\left(\frac{1.841}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2 = \omega^2\mu\epsilon \quad (21)$$

hence

$$f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{1.841}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2} \quad (22)$$