# Lecture 4c

## *Cylindrical waveguide*

### Introduction

In previous lectures we examined the fields in a rectangular waveguide. In any coordinate system which keeps the $z$ coordinate of rectangular coordinates (any type of "cylindrical coordinates") it is natural to examine waveguides with the type of cross section appropriate to that coordinate system. These fields will still have $z$ dependence of the form $e^{-j\beta_z z}$, and the ideas of cutoff frequency and waveguide dispersion will carry over from the rectangular case.

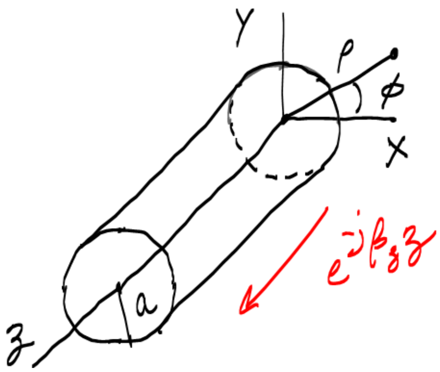


*Figure 1: Cylindrical waveguide geometry.*

In this lecture we consider cylindrical waveguides with circular cross sections and PEC surfaces as illustrated above.

### TE$^z$ modes

In cylindrical coordinates, a general $\mathrm{TE}^z$ mode which propagates in the $z$ direction is described by

$$F_z(\rho,\phi,z)=\begin{Bmatrix} J_m(\beta_\rho \rho) \\ Y_m(\beta_\rho \rho)\end{Bmatrix}\begin{Bmatrix}\cos(m\phi) \\ \sin(m\phi)\end{Bmatrix} e^{-j\beta_z z} \tag{1}$$

where the brace notation represents an arbitrary linear combination of the corresponding functions. In order for the Helmholtz equation to be satisfied we have the condition

$$\beta_\rho^2+\beta_z^2=\beta^2=\omega^2\mu\epsilon \tag{2}$$

where $\mu,\epsilon$ are the permeability and permittivity of the material inside the waveguide.

In the rectangular waveguide we had boundary conditions at the four surfaces $x=0,a$, $y=0,b$. For the cylindrical waveguide the limits on $\rho$ are $0\le\rho\le a$, and the limits on $\phi$ are $0\le\phi\le 2\pi$. What are the corresponding boundary conditions? Only one of the "boundaries" $\rho=0,a$, $\phi=0,2\pi$ contains a PEC surface. The boundary conditions at $\rho=a$ are that the tangential components of **E** vanish. As illustrated in the following figure, since this is a $\mathrm{TE}^z$ field $E_z\equiv 0$ and the only tangential component is $E_\phi$.

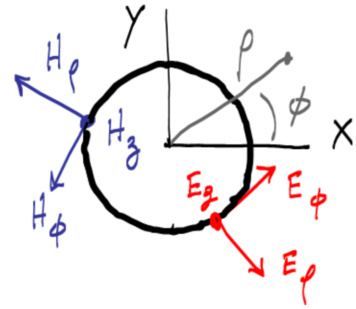


*Figure 2: Boundary conditions for cylindrical waveguide.*

What about the "boundary" $\rho=0$? Although the field need not be zero there it must be finite, and that does constrain the solution since the $Y_m$ functions blow up at the origin. The $\phi=0,2\pi$ "boundaries" represent the same point in space. Therefore, the fields at $\phi=2\pi$ must be identical to the fields at $\phi=0$. In other words, the fields must be periodic in $\phi$ with period $2\pi$. This will put constraints on the $\phi$ functions.

Our complete "boundary conditions" read

$$\begin{array}{l} \text{fields have period } 2\pi \text{ in } \phi \\ \text{fields are finite at } \rho=0 \\ E_\phi=E_z=0 \quad \text{at} \quad \rho=a \end{array} \tag{3}$$

For the function $\cos(m\phi)$ and $\sin(m\phi)$ to have period $2\pi$ we must require that $m$ be an integer. Other than that we can take any combination of the cosine and sine. Since $m$ is an integer, the Bessel functions have to be of integer order. Since $|Y_m(0)|=\infty$ the second condition requires that we use only the $J_m(\beta_\rho\rho)$ functions in our solution. If we write a general linear combination of the cosine and sine in the form

$$a\cos(m\phi)+b\sin(m\phi)=F_0\cos(m[\phi-\phi_0]) \tag{4}$$

where

$$\begin{aligned} a&=F_0\cos(m\phi_0) \\ b&=F_0\sin(m\phi_0)\end{aligned} \tag{5}$$

then our potential will have the form

$$F_z(\rho,\phi,z)=F_0 J_m(\beta_\rho\rho)\cos(m[\phi-\phi_0])e^{-j\beta_z z} \tag{6}$$

The third condition requires that the expression

$$E_\phi=\frac{1}{\epsilon}\frac{\partial}{\partial\rho}F_z \tag{7}$$

vanish at $\rho=a$. Since the $\rho$ dependence is through the factor $J_m(\beta_\rho\rho)$, $E_\phi$ will vanish if

$$\frac{d}{d\rho} J_m(\beta_\rho \rho) = \beta_\rho J_m'(\beta_\rho \rho) = 0 \tag{8}$$

Our boundary condition at $\rho = a$ is therefore

$$J_m'(\beta_\rho a) = 0 \tag{9}$$

If we use $x_{mn}'$ to denote the $n^{\text{th}}$ zero of $J_m'(x)$ (with $n = 1,2,\ldots$), then the allowable values for $\beta_\rho$ are

$$\beta_\rho = \frac{x_{mn}'}{a} \tag{10}$$

The zeros of some Bessel functions and their derivatives are illustrated in the following figure.

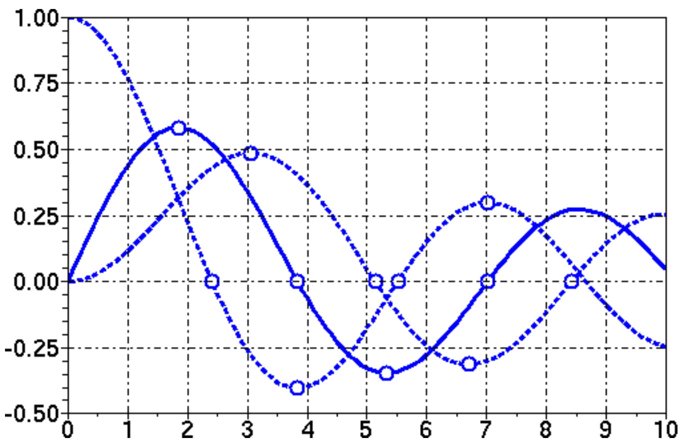


*Figure 3: $J_m(x)$ for $m = 0,1,2$. Circles locate zeros of the functions and their derivatives.*

Numerically we find

| function | first zero | second zero |
|---|---|---|
| $J_0(x)$ | 2.405 | 5.520 |
| $J_0'(x)$ | 3.832 | 7.016 |
| $J_1(x)$ | 3.832 | 7.016 |
| $J_1'(x)$ | 1.841 | 5.331 |
| $J_2(x)$ | 5.136 | 8.417 |
| $J_2'(x)$ | 3.054 | 6.706 |

(11)

With $\beta_\rho = x_{mn}'/a$ where $x_{mn}'$ is the $n^{\text{th}}$ zero of $J_m'(x)$, our solution is

$$F_z(\rho,\phi,z) = F_0 J_m\left(\beta_\rho \rho\right) \cos(m[\phi - \phi_0]) e^{-j\beta_z z} \tag{12}$$

with

$$\beta_z = \sqrt{\omega^2 \mu \epsilon - \beta_\rho^2} \tag{13}$$

We refer to this as the $\text{TE}^z_{mn}$ mode.

The resulting electric field components are

$$\begin{aligned} E_\rho &= \frac{m}{\epsilon \rho} F_0 J_m(\beta_\rho \rho) \sin(m[\phi - \phi_0]) e^{-j\beta_z z} \\ E_\phi &= \frac{\beta_\rho}{\epsilon} F_0 J_m'(\beta_\rho \rho) \cos(m[\phi - \phi_0]) e^{-j\beta_z z} \\ E_z &= 0 \end{aligned} \tag{14}$$

and the magnetic field components are

$$\begin{aligned} H_\rho &= -\frac{\beta_\rho \beta_z}{\omega \mu \epsilon} F_0 J_m'(\beta_\rho \rho) \cos(m[\phi - \phi_0]) e^{-j\beta_z z} \\ H_\phi &= \frac{m \beta_z}{\omega \mu \epsilon \rho} F_0 J_m(\beta_\rho \rho) \sin(m[\phi - \phi_0]) e^{-j\beta_z z} \\ H_z &= -j \frac{\beta_\rho^2}{\omega \mu \epsilon} F_0 J_m(\beta_\rho \rho) \cos(m[\phi - \phi_0]) e^{-j\beta_z z} \end{aligned} \tag{15}$$

Notice that the ratio of orthogonal **E** and **H** components is a constant

$$\frac{E_\rho}{H_\phi} = \frac{E_\phi}{-H_\rho} = \frac{\omega \mu}{\beta_z} \tag{16}$$

Therefore the wave impedance for a $\text{TE}^z$ mode is

$$Z = \frac{\omega \mu}{\beta_z} \tag{17}$$

## TM$^z$ modes

A general $\text{TM}^z_{mn}$ mode which propagates in the *z* direction and is finite at the origin is described by

$$A_z(\rho,\phi,z) = A_0 J_m(\beta_\rho \rho) \cos(m[\phi - \phi_0]) e^{-j\beta_z z} \tag{18}$$

The tangential components of **E** at $\rho = a$ are

$$\begin{aligned} E_\phi &= -\frac{j}{\omega \mu \epsilon \rho} \frac{\partial}{\partial \phi} \frac{\partial}{\partial z} A_z \\ E_z &= -\frac{j}{\omega \mu \epsilon} \left[ \beta^2 A_z + \frac{\partial^2}{\partial z^2} A_z \right] \end{aligned} \tag{19}$$

Neither of these expressions involves a $\rho$ derivative, so both $E_\phi, E_z$ will contain a factor $J_m(\beta_\rho \rho)$. The boundary condition at the PEC surface $\rho = a$ will therefore be satisfied if

$$J_m(\beta_\rho a) = 0 \tag{20}$$

If we use $x_{mn}$ to denote the $n^{\text{th}}$ zero of $J_m(x)$ (with $n = 1,2,\ldots$), then the allowable values for $\beta_\rho$ are

$$\beta_\rho = \frac{x_{mn}}{a} \tag{21}$$

and our solution is

$$F_z(\rho,\phi,z) = F_0 J_m\left(\beta_\rho \rho\right) \cos(m[\phi - \phi_0]) e^{-j\beta_z z} \tag{22}$$

with

$$\beta_z = \sqrt{\omega^2 \mu \epsilon - \beta_\rho^2} \tag{23}$$

The resulting magnetic field components are

$$\begin{aligned} H_\rho &= -\frac{m}{\mu \rho} A_0 J_m(\beta_\rho \rho) \sin(m[\phi - \phi_0]) e^{-j\beta_z z} \\ H_\phi &= -\frac{\beta_\rho}{\mu} A_0 J_m'(\beta_\rho \rho) \cos(m[\phi - \phi_0]) e^{-j\beta_z z} \\ H_z &= 0 \end{aligned} \tag{24}$$

and the electric field components are

$$\begin{aligned} E_\rho &= -\frac{\beta_\rho \beta_z}{\omega \mu \epsilon} A_0 J_m'(\beta_\rho \rho) \cos(m[\phi - \phi_0]) e^{-j\beta_z z} \\ E_\phi &= \frac{m \beta_z}{\omega \mu \epsilon \rho} A_0 J_m(\beta_\rho \rho) \sin(m[\phi - \phi_0]) e^{-j\beta_z z} \\ E_z &= -j \frac{\beta_\rho^2}{\omega \mu \epsilon} A_0 J_m(\beta_\rho \rho) \cos(m[\phi - \phi_0]) e^{-j\beta_z z} \end{aligned} \tag{25}$$

Notice that the ratio of orthogonal **E** and **H** components is a constant

$$\frac{E_\rho}{H_\phi} = \frac{E_\phi}{-H_\rho} = \frac{\beta_z}{\omega \epsilon} \tag{26}$$

Therefore the wave impedance for a $\mathrm{TM}^z$ mode is

$$Z = \frac{\beta_z}{\omega \epsilon} \tag{27}$$

## Cutoff

When $\beta_z = 0$ the mode no longer propagates along the *z* axis. This defines the *cutoff frequency*

$$f_c = \frac{1}{2\pi} \frac{\beta_\rho}{\sqrt{\mu \epsilon}} \tag{28}$$

Below the cutoff frequency $\beta_z$ becomes imaginary and the field decays exponentially as $e^{-\alpha_z z}$ with

$$\alpha_z = \sqrt{\beta_\rho^2 - \omega^2 \mu \epsilon} \tag{29}$$

From (11) we see that the lowest cutoff frequency will be that of the $\mathrm{TE}_{11}^z$ mode

$$f_c = \frac{1}{2\pi} \frac{1.841}{a\sqrt{\mu \epsilon}} = \frac{0.293}{a\sqrt{\mu \epsilon}} \tag{30}$$

The next lowest cutoff frequency will be that of the $\mathrm{TM}_{01}^z$ mode

$$f_c = \frac{1}{2\pi} \frac{2.405}{a\sqrt{\mu \epsilon}} = \frac{0.383}{a\sqrt{\mu \epsilon}} \tag{31}$$

The $\mathrm{TE}_{11}^z$ is therefore the dominant mode and the waveguide has single-mode operation over the frequency range

$$\frac{0.293}{a\sqrt{\mu \epsilon}} < f < \frac{0.383}{a\sqrt{\mu \epsilon}} \tag{32}$$

## Dominant mode

The dominant $\mathrm{TE}_{11}^z$ mode has

$$F_z(\rho, \phi, z) = F_0 J_1(\beta_\rho \rho) \cos(\phi - \phi_0) e^{-j\beta_z z} \tag{33}$$

with

$$\beta_\rho = \frac{1.841}{a} \tag{34}$$

and

$$\beta_z = \sqrt{\omega^2 \mu \epsilon - \beta_\rho^2} \tag{35}$$

The electric field components are

$$\begin{aligned} E_\rho &= \frac{1}{\epsilon \rho} F_0 J_1(\beta_\rho \rho) \sin(\phi - \phi_0) e^{-j\beta_z z} \\ E_\phi &= \frac{\beta_\rho}{\epsilon} F_0 J_1'(\beta_\rho \rho) \cos(\phi - \phi_0) e^{-j\beta_z z} \\ E_z &= 0 \end{aligned} \tag{36}$$

and the magnetic field components are

$$\begin{aligned} H_\rho &= -\frac{\beta_\rho \beta_z}{\omega \mu \epsilon} F_0 J_1'(\beta_\rho \rho) \cos(\phi - \phi_0) e^{-j\beta_z z} \\ H_\phi &= \frac{\beta_z}{\omega \mu \epsilon \rho} F_0 J_1(\beta_\rho \rho) \sin(\phi - \phi_0) e^{-j\beta_z z} \\ H_\rho &= -j \frac{\beta_\rho^2}{\omega \mu \epsilon} F_0 J_1(\beta_\rho \rho) \cos(\phi - \phi_0) e^{-j\beta_z z} \end{aligned} \tag{37}$$

The Bessel function derivative can be conveniently calculated using the formula

$$J_1'(x) = \frac{1}{2}\left[J_0(x) - J_2(x)\right] \tag{38}$$

## The most general waveguide field

For the rectangular waveguide we saw that an arbitrary field expanded in Fourier series corresponded to a superposition of the waveguide modes. Therefore, any field that could exist in the waveguide could be expanded in terms of the waveguide modes. For the cylindrical waveguide we no longer have only sines and cosines but also Bessel functions. Let's see if our modes still form a "complete orthogonal set."

Consider a general $\mathrm{TM}^z$ field described by $A_z(\rho, \phi, z)$. In the plane $z = 0$ and for a particular value $\rho = \rho_0$ this is a function of $\phi$ alone. We can therefore expand it in a Fourier series

$$\begin{aligned} A_z(\rho_{0,} \phi, 0) &= a_0 + \sum_{m=1}^{\infty} a_m \cos m\phi + b_m \sin m\phi \\ &= \sum_{m=0}^{\infty} A_m \cos(m[\phi - \phi_m]) \end{aligned} \tag{39}$$

Either form works and they are related by (5).

Now, if we repeat this process for different values of $\rho$ we will get Fourier coefficients that are functions of $\rho$. The formulas are

$$a_0(\rho)=\frac{1}{2\pi}\int_0^{2\pi} F_z(\rho,\phi,0)\,d\phi$$
$$a_m(\rho)=\frac{1}{\pi}\int_0^{2\pi} F_z(\rho,\phi,0)\cos(m\phi)\,d\phi \tag{40}$$
$$b_m(\rho)=\frac{1}{\pi}\int_0^{2\pi} F_z(\rho,\phi,0)\sin(m\phi)\,d\phi$$

We would now like to expand these coefficient functions in series that correspond to the $\rho$ dependence of our $\mathrm{TE}^z$ modes. But this involves Bessel functions instead of cosines and sines. Can we do this? The answer is "yes" and the reason is Sturm-Liouville theory. The Bessel functions $J_m(\beta_\rho\rho)$ satisfy the differential equation

$$f''+\frac{1}{\rho}f'+\left(\beta_\rho^2-\frac{m^2}{\rho^2}\right)f=0 \tag{41}$$

This can easily be put into the Sturm-Liouville form

$$[u(\rho)f']'+[v(\rho)+\lambda w(\rho)]f=0 \tag{42}$$

to get

$$(\rho f')'+\left(\lambda\rho-\frac{m^2}{\rho}\right)f=0 \tag{43}$$

We have $\lambda=\beta_\rho^2$, $u(\rho)=w(\rho)=\rho$ and $v(\rho)=-m^2/\rho$. As discussed in Lecture 1d, we know that the eigenfunctions of a Sturm-Liouville problem form a complete, orthogonal set of functions for the corresponding boundary conditions. Therefore we know that the Bessel function can be used as the basis for a generalized Fourier series of the form

$$f(\rho)=\sum_{n=1}^{\infty} c_n J_m(\beta_{\rho mn}\rho) \tag{44}$$

where $\beta_{\rho mn}=x_{mn}/a$ and $m$ is any fixed integer. This is often called a *Fourier-Bessel series*. Note that we use only a single order of Bessel function. What varies with $n$ is $\beta_{\rho mn}$. This idea is illustrated in the following figure for the case of $J_\pi(x)$.

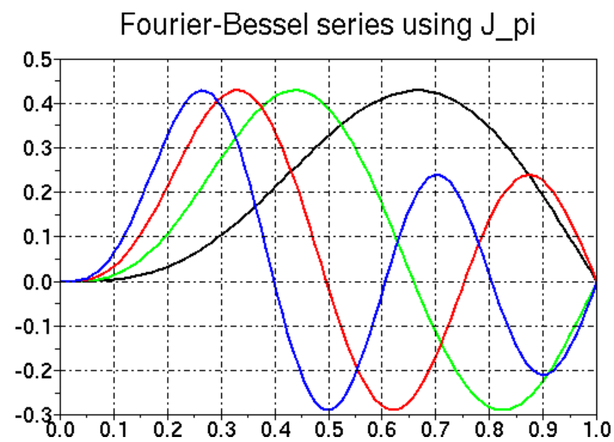


*Figure 4: The functions $J_\pi(x\cdot x_{\pi n})$ where $x_{\pi n}$ is the n-th zero of $J_\pi(x)$. These can be used as the basis of a generalized Fourier series over the interval [0,1] in the same way the functions $\sin(n\pi x)$ can be.*

The coefficients are

$$c_n=\frac{\int_0^a \rho f(\rho)J_m(\beta_{\rho mn}\rho)\,d\rho}{\int_0^a \rho J_m^2(\beta_{\rho mn}\rho)\,d\rho} \tag{45}$$

Applying this idea to the $a_m(\rho), b_m(\rho)$ functions, for example

$$a_m(\rho)=\sum_{n=1}^{\infty} c_{mn} J_m(\beta_{\rho mn}\rho) \tag{46}$$

and using (5) we will end up with

$$A_z(\rho,\phi,0) =\sum_m\sum_n A_{mn}J_m\left(\beta_{\rho mn}\rho\right)\cos(m[\phi-\phi_{mn}]) \tag{47}$$

Identifying this as a superposition of $\mathrm{TM}^z_{mn}$ modes, for other values of $z$ we can write

$$A_z(\rho,\phi,z) =\sum_m\sum_n A_{mn}J_m\left(\beta_{\rho mn}\rho\right)\cos(m[\phi-\phi_{mn}])e^{-j\beta_{zmn}z} \tag{48}$$

where

$$\beta_{zmn}=\sqrt{\omega^2\mu\epsilon-\beta_{\rho mn}^2} \tag{49}$$

Therefore, as for the rectangular waveguide, *any* $\mathrm{TM}^z$ field that can exist inside the cylindrical waveguide can be represented as a superposition of our modes. By a very similar process we can say the same thing about any $\mathrm{TE}^z$ field. Combining these we can represent any solution to Maxwell's equations that satisfies the waveguide boundary conditions as a superposition of our modes. In this sense they tell us everything we need to know about the waveguide.