Lecture 4b

Bessel functions

Introduction

In the previous lecture the separation of variables method led to *Bessel's equation*

$$y''+\frac{1}{x}y'+\left(1-\frac{\nu^2}{x^2}\right)y=0 \tag{1}$$

Here we use ν instead of m to emphasize the complete generality of the separation constant (arbitrary complex number). Our goal in this lecture is to obtain the general solution of this equation. The resulting *Bessel functions* are among the most commonly encountered so-called *special functions*. They naturally arise in problems in cylindrical coordinates and so are sometimes called *cylinder functions*. However, we will see that they also come up in spherical coordinates and other applications.

Before attempting a rigorous solution, let's examine the qualitative behavior of the solutions to this equation. When x gets very large $1-\nu^2/x^2\to 1$ so the functions will approximately satisfy

$$y''+\frac{1}{x}y'+y=0 \tag{2}$$

This is independent of the parameter ν. Moreover if we neglect the y'/x term (not rigorously valid, but we are just "getting a feel") we have $y''+y=0$. The solutions to this are simply $\cos(x),\sin(x)$. It appears that as $x\to\infty$ the solutions for any value of ν will oscillate like some combination of $\cos(x),\sin(x)$.

On the other hand, when x gets very small $1-\nu^2/x^2\to-\nu^2/x^2$ and the functions will approximately satisfy

$$y''+\frac{1}{x}y'-\frac{\nu^2}{x^2}y=0 \tag{3}$$

The solutions to this are $y=x^{\nu},x^{-\nu}$ when $\nu\neq 0$ and $y=1,\ln(x)$ when $\nu=0$. So when $x\to 0$ it appears that one solution will go to zero as x^{ν} while the other will blow up like $x^{-\nu}$. For the special case $\nu=0$, one solution will approach a constant value while the other will blow up like $\ln(x)$.

Generalized factorial function

In developing our solutions we will need the *generalized factorial function*. This is

$$\nu!=\int_0^{\infty}t^{\nu}e^{-t}\,dt \tag{4}$$

Using integration by parts it is easy to show that

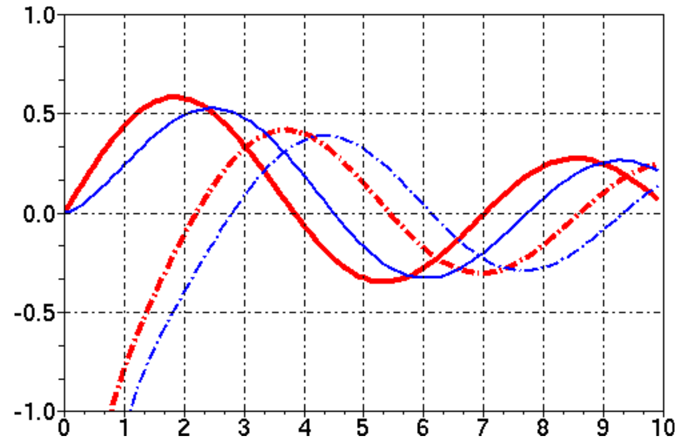

Figure 1: Example solutions to Bessel's equation. Thick (red) curves have $\nu=1$. Thin (blue) curves have $\nu=1.5$. Solid curves display x^{ν} behavior at $x\to 0$ while dotted curves display $x^{-\nu}$ behavior.

$$(\nu+1)!=(\nu+1)\nu! \tag{5}$$

Since

$$\int_0^{\infty}e^{-t}\,dt=1=0! \tag{6}$$

we have

$$m!=\int_0^{\infty}t^m e^{-t}\,dt \tag{7}$$

for $m=0,1,2,\ldots$ where $m!=m(m-1)(m-2)\cdots 1$ is the regular integer factorial. Alternately we can use the *gamma function* and write

$$\nu!=\Gamma(\nu+1) \tag{8}$$

where

$$\Gamma(\nu)=\int_0^{\infty}t^{\nu-1}e^{-t}\,dt \tag{9}$$

Most math packages will evaluate either the generalized factorial (e.g., Maxima) or the gamma function (e.g., Scilab). The generalized factorial function is well-defined for all real numbers except the negative integers where it "blows up." Indeed

$$0!=0(-1)!=1 \tag{10}$$

implies $(-1)!=\infty$. Then $(-1)!=-1(-2)!$ etc. require that $(-m)!=\pm\infty$ for $m=1,2,\ldots$. A plot of the factorial function is given in the following figure.

A useful value in some applications is $(1/2)!=\sqrt{\pi}/2$. An approximation valid for large values is the *Stirling formula*

$$\nu!\sim\sqrt{2\pi}\,\nu^{\nu+1/2}e^{-\nu} \tag{11}$$

For $\nu \geq 8$ this is good to within 1%.

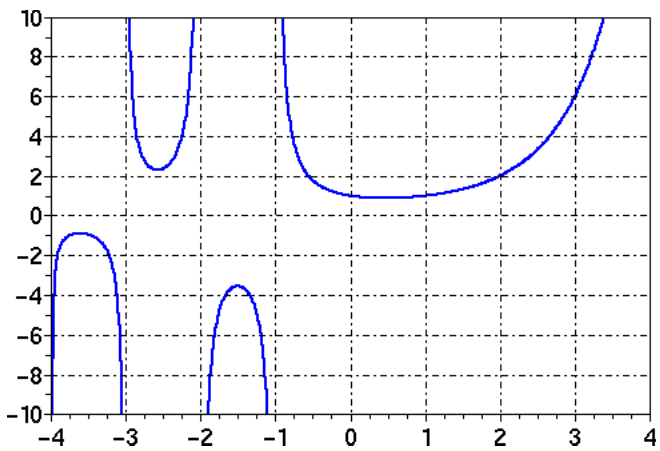

Figure 2: Generalized factorial function x!

Bessel functions of the 1st kind

Now let's solve the Bessel equation (1). The functions $p(x)=1/x$ and $q(x)=1-\nu^2/x^2$ are singular at $x=0$ but $x\,p(x), x^2\,q(x)$ are analytic, so we need to use the Frobenius method. We look for solutions of the form

$$y(x)=x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n} \tag{12}$$

where by assumption $a_0 \neq 0$. It is convenient to multiply (1) through by x^2 to obtain

$$x^2\,y''+x\,y'+\left(x^2-\nu^2\right)y=0 \tag{13}$$

Substituting the series and

$$\begin{aligned} y'(x)&=\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1} \\ y''(x)&=\sum_{n=0}^{\infty}(r+n)(r+n-1)\,a_n x^{r+n-2} \end{aligned} \tag{14}$$

into the differential equation we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty}\left[(r+n)(r+n-1)\,a_n+(r+n)a_n-\nu^2 a_n\right]x^{r+n} \\ &\qquad +\sum_{n=0}^{\infty} a_n x^{r+n+2}=0 \end{aligned} \tag{15}$$

Since

$$(r+n)(r+n-1)+(r+n)-\nu^2=(r+n)^2-\nu^2 \tag{16}$$

we can write this as

$$\sum_{n=0}^{\infty}[(r+n)^2-\nu^2]a_n x^{r+n}+\sum_{n=2}^{\infty} a_{n-2} x^{r+n}=0 \tag{17}$$

The coefficient of the x^r term is

$$(r^2-\nu^2)\,a_0=0$$

This requires

$$r=\pm\nu \tag{18}$$

This will give us two solutions with the $x\to 0$ behavior $y=x^{\nu}, x^{-\nu}$ that we would expect from our initial analysis. If $\nu-(-\nu)=2\nu$ is not an integer, then these are guaranteed to be linearly independent. That is, ν should not be integer or half integer. (More on this later.)

The coefficient of the x^{r+1} term is

$$[(r+1)^2-\nu^2]\,a_1=0 \tag{19}$$

Except for the special case $r^2=(r+1)^2$ which requires $r=-1/2$ (more on that later) we must have

$$a_1=0 \tag{20}$$

The coefficient of the x^{r+n} term is

$$[(n\pm\nu)^2-\nu^2]\,a_n+a_{n-2}=0 \tag{21}$$

This requires

$$a_n=-\frac{a_{n-2}}{(n\pm\nu)^2-\nu^2} \tag{22}$$

We see that $a_1=a_3=a_5=\cdots=0$ so all odd-numbered coefficients are zero. Since

$$(n\pm\nu)^2-\nu^2=n^2\pm 2n\,\nu=n(n\pm 2\nu) \tag{23}$$

if we call $n=2k$ we can write

$$a_{2k}=-\frac{a_{2k-2}}{4k(k\pm\nu)} \tag{24}$$

for our coefficient recursion. Of the two solutions $r=\pm\nu$ let's start with $r=\nu$. The first coefficient is arbitrary. Let's take

$$a_0=\frac{1}{2^{\nu}\,\nu!} \tag{25}$$

then

$$\begin{aligned} a_2&=-\frac{1}{2^{\nu}\,\nu!}\,\frac{1}{4(1)(\nu+1)} \\ a_4&=\frac{1}{2^{\nu}\,\nu!}\,\frac{1}{4(1)(\nu+1)}\,\frac{1}{4(2)(\nu+2)} \end{aligned} \tag{26}$$

and, since

$$\begin{aligned} &4\cdot 4=2^{2\cdot 2} \\ &(1)(2)=2! \\ &\nu!(\nu+1)(\nu+2)=(\nu+2)! \end{aligned} \tag{27}$$

we will have

$$a_{2k}=(-1)^k\,\frac{1}{k!(\nu+k)!}\,\frac{1}{2^{\nu}\,2^{2k}} \tag{28}$$

Our solution is therefore

$$J_\nu(x)=\left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+\nu)!}\left(\frac{x}{2}\right)^{2k} \tag{29}$$

We call this the *Bessel function of the first kind of order* ν. The process with $r=-\nu$ results in

$$J_{-\nu}(x)=\left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k-\nu)!}\left(\frac{x}{2}\right)^{2k} \tag{30}$$

Provided ν is not an integer, a general solution of (1) is

$$y(x)=a\,J_\nu(x)+b\,J_{-\nu}(x) \tag{31}$$

(We'll see below that half-integer ν values present no problem.) Since

$$\begin{aligned}&\left[\frac{1}{(k+1)!(k+1+\nu)!}\left(\frac{x}{2}\right)^{2(k+1)}\right]\left[\frac{1}{(k)!(k+\nu)!}\left(\frac{x}{2}\right)^{2k}\right]^{-1}\\ &=\frac{(x/2)^2}{(k+1)(k+1+\nu)}\to 0\end{aligned} \tag{32}$$

as $k\to\infty$ for any fixed *x*, we have that by the ratio test our series solution converges for all values of *x*.

Bessel functions of the second kind

In the Frobenius method, if the two *r* values differ by an integer then our second solution may not be linearly independent. If $r=\nu=m$ for non-negative integer *m* then one solution is

$$J_m(x)=\left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+m)!}\left(\frac{x}{2}\right)^{2k} \tag{33}$$

Our other solution is (formally)

$$J_{-m}(x)=\left(\frac{x}{2}\right)^{-m} \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k-m)!}\left(\frac{x}{2}\right)^{2k} \tag{34}$$

However $(k-m)$ will take on negative integer values when $k<m$. Since $|(k-m)!|=\infty$ in those cases the corresponding terms will vanish. We are left with

$$J_{-m}(x)=\left(\frac{x}{2}\right)^{-m} \sum_{k=m}^{\infty}\frac{(-1)^k}{k!(k-m)!}\left(\frac{x}{2}\right)^{2k} \tag{35}$$

Calling $k=n+m$ we can write this as

$$\begin{aligned}J_{-m}(x)&=\left(\frac{x}{2}\right)^{-m} \sum_{n=0}^{\infty}\frac{(-1)^{(n+m)}}{(n+m)!n!}\left(\frac{x}{2}\right)^{2n+2m}\\ &=(-1)^m\left(\frac{x}{2}\right)^{m} \sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!(n+m)!}\left(\frac{x}{2}\right)^{2n}\\ &=(-1)^m J_m(x)\end{aligned} \tag{36}$$

Therefore $J_{-m}(x)$ is clearly not linearly independent of $J_m(x)$ and we need to find another solution for the integer-order case.

Following the Frobenius method, the second solution will have the form

$$Y_m(x)=\ln(x)\,J_m(x)+x^{-m}\sum_{k=0}^{\infty}b_k x^k \tag{37}$$

Writing this as $Y_m(x)=\ln(x)\,J_m(x)+u(x)$ and plugging it into the differential equation

$$x^2 Y_m''+x\,Y_m'+\left(x^2-m^2\right)Y_m=0 \tag{38}$$

results in

$$x^2 u''+x\,u'+\left(x^2-m^2\right)u=-2x\,J_m'(x) \tag{39}$$

for the inhomogeneous differential equation satisfied by

$$u=x^{-m}\sum_{k=0}^{\infty}b_k x^k \tag{40}$$

We can put this series into the LODE and solve for the coefficients b_k and obtain the second solution for the integer-order case. However, it is awkward to use $J_\nu(x), J_{-\nu}(x)$ as solutions for ν not an integer and then have to switch to $J_m(x), Y_m(x)$ for ν an integer. A way around this is to use the Bessel function of the first kind, $J_\nu(x)$, as one solution and then to define the *Bessel function of the second kind* as

$$Y_\nu(x)\equiv\frac{J_\nu(x)\cos\nu\pi-J_{-\nu}(x)}{\sin\nu\pi} \tag{41}$$

for any value of ν. If ν is not an integer then this is simply a linear combination of $J_\nu(x), J_{-\nu}(x)$, so it solves the Bessel equation and is linearly independent of $J_\nu(x)$. For ν an integer it becomes

$$Y_m(x)=\frac{J_m(x)(-1)^m-J_{-m}(x)}{\sin m\pi}=\frac{0}{0} \tag{42}$$

so we need to take the limit. We can do using L'Hospital's rule. We obtain

$$\begin{aligned}Y_m(x)&=\lim_{\nu\to m}\left[\frac{\frac{\partial}{\partial\nu}\left[J_\nu(x)\cos\nu\pi-J_{-\nu}(x)\right]}{\frac{\partial}{\partial\nu}\sin\nu\pi}\right]\\ &=\lim_{\nu\to m}\frac{\frac{\partial J_\nu(x)}{\partial\nu}-(-1)^m\frac{\partial J_{-\nu}(x)}{\partial\nu}}{\pi}\end{aligned} \tag{43}$$

Since $J_\nu(x)$ contains a factor of x^ν and $\partial x^\nu/\partial\nu=x^\nu\ln(x)$ we see that $Y_m(x)$ will contain the $\ln(x)$ behavior that we would get by using the Frobenious method. Taking derivatives with respect to ν of the $J_\nu(x), J_{-\nu}(x)$ series is complicated, but doable. Following through with the calculation one obtains

$$\begin{aligned} Y_m(x) &= \frac{2}{\pi}\left(\gamma + \ln\frac{x}{2}\right) J_m(x) \\ &-\frac{1}{\pi}\left(\frac{x}{2}\right)^{-m} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k} \\ &-\frac{1}{\pi}\left(\frac{x}{2}\right)^{m} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!}(h_k + h_{k+m})\left(\frac{x}{2}\right)^{2k} \end{aligned} \tag{44}$$

where $h_0 = 0$, $h_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$ and $\gamma = 0.5772\ldots$ is *Euler's constant*.

Fourier series and integral forms

Bessel functions also arise in certain Fourier series, and this leads to useful *integral forms* for the functions. Consider the function $\cos(x \sin\theta)$ which comes up in some antenna problems, among others places. Applying the Taylor series for $\cos x$ to this function we have

$$\cos(x\sin\theta) = 1 - \frac{(x\sin\theta)^2}{2!} + \frac{(x\sin\theta)^4}{4!} - \frac{(x\sin\theta)^6}{6!} + \cdots \tag{45}$$

Now

$$\begin{aligned} \sin^2\theta &= \frac{1-\cos 2\theta}{2} \\ \sin^4\theta &= \frac{3 - 4\cos 2\theta + \cos 4\theta}{8} \\ \sin^6\theta &= \frac{10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta}{32} \end{aligned} \tag{46}$$

and so on. Collecting all terms with the same θ dependence we obtain

$$\begin{aligned} \cos(x\sin\theta) &= \left[1 - \left(\frac{1}{2}\right)\frac{x^2}{2!} + \left(\frac{3}{8}\right)\frac{x^4}{4!} - \left(\frac{10}{32}\right)\frac{x^6}{6!} + \cdots\right] \\ &+ \cos 2\theta\left[\left(\frac{1}{2}\right)\frac{x^2}{2!} - \left(\frac{4}{8}\right)\frac{x^4}{4!} + \left(\frac{15}{32}\right)\frac{x^6}{6!} + \cdots\right] \\ &+ \cos 4\theta\left[\left(\frac{1}{8}\right)\frac{x^4}{4!} - \left(\frac{6}{32}\right)\frac{x^6}{6!} + \cdots\right] \\ &+ \cdots \end{aligned} \tag{47}$$

This is a Fourier series for $\cos(x\sin\theta)$. The "coefficients" are functions of *x* which correspond to the power series of various Bessel functions as can be verified by comparing with (33)

$$\begin{aligned} &\cos(x\sin\theta) = \\ &\quad J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots \end{aligned} \tag{48}$$

Likewise, one finds

$$\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \cdots \tag{49}$$

Applying the normal method of calculating Fourier coefficients we can therefore write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(x\sin\theta)\cos m\theta \, d\theta = \begin{cases} J_m(x) & m \text{ even} \\ 0 & m \text{ odd} \end{cases} \tag{50}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(x\sin\theta)\sin m\theta \, d\theta = \begin{cases} 0 & m \text{ even} \\ J_m(x) & m \text{ odd} \end{cases} \tag{51}$$

Finally, since

$$\cos(x\sin\theta)\cos m\theta + \sin(x\sin\theta)\sin m\theta = \cos(x\sin\theta - m\theta) \tag{52}$$

we have the integral form of $J_m(x)$

$$J_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(x\sin\theta - m\theta)\, d\theta \tag{53}$$

Equivalent forms are

$$J_m(x) = \frac{1}{\pi}\int_{0}^{\pi} \cos(x\sin\theta - m\theta)\, d\theta \tag{54}$$

and

$$J_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(x\sin\theta - m\theta)}\, d\theta \tag{55}$$

These are valid only for non-negative integer values *m*. Using complex contour integration more general integral forms can be found (see the Carrier reference for details). For arbitrary ν one finds

$$\begin{aligned} J_\nu(x) &= \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - \nu\theta)\, d\theta \\ &-\frac{\sin\nu\pi}{\pi}\int_0^{\infty} e^{-(x\sinh\phi + \nu\phi)}\, d\phi \end{aligned} \tag{56}$$

and

$$\begin{aligned} Y_\nu(x) &= \frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta - \nu\theta)\, d\theta \\ &-\frac{1}{\pi}\int_0^{\infty} (e^{\nu\phi} + e^{-\nu\phi}\cos\nu\pi) e^{-x\sinh\phi}\, d\phi \end{aligned} \tag{57}$$

Note that the first of these reduces to our result for $\nu = m$ an integer.

Asymptotic behavior

Asymptotic behavior in the limits $x \to 0$, $x \to \infty$ is important in many applications. The $x \to 0$ forms follow directly by taking the first terms of the series solutions. We have

$$\begin{aligned} J_0(x) &\sim 1 & Y_0(x) &\sim \frac{2}{\pi}\left(\gamma + \ln\frac{x}{2}\right) \\ J_\nu(x) &\sim \frac{1}{\nu!}\left(\frac{x}{2}\right)^{\nu} & Y_\nu(x) &\sim -\frac{(\nu-1)!}{\pi}\left(\frac{x}{2}\right)^{-\nu} \end{aligned} \tag{58}$$

with $\gamma = 0.5772\ldots$.

Now let's treat the $x\to\infty$ limit. It's instructive to begin by considering the substitution $y=u/\sqrt{(x)}$ as it transforms (1) into

$$u''+\left(1-\frac{\nu^2-1/4}{x^2}\right)u=0 \tag{59}$$

When $\nu=1/2$ this becomes simply $u''+u=0$ which has solutions $\sin x, \cos x$. So for $\nu=1/2$ Bessel's equation is *exactly* solved by $\sin(x)/\sqrt{x}$ and $\cos(x)/\sqrt{x}$. Multiplying by constants to get the $x\to 0$ behavior of (58) we obtain

$$\begin{aligned} J_{1/2}(x) &= \sqrt{\frac{2}{\pi x}}\sin(x) \\ Y_{1/2}(x) &= -\sqrt{\frac{2}{\pi x}}\cos(x) \end{aligned} \tag{60}$$

Keep in mind that these are not asymptotic expressions, they are exact for all *x*. We see that at least some of the Bessel functions are quite simple! Similarly one can show that $J_{-1/2}(x)=-Y_{1/2}(x)$ and $Y_{-1/2}(x)=J_{1/2}(x)$.

For arbitrary values of ν "asymptotic analysis" provides the $x\to\infty$ asymptotic forms (see Carrier for details)

$$\begin{aligned} J_\nu(x) &\sim \sqrt{\frac{2}{\pi x}}\cos\left(x-[2\nu+1]\frac{\pi}{4}\right) \\ Y_\nu(x) &\sim \sqrt{\frac{2}{\pi x}}\sin\left(x-[2\nu+1]\frac{\pi}{4}\right) \end{aligned} \tag{61}$$

Unlike the $\nu=1/2$ case, these are in general only valid for $x\to\infty$.

Recursion and derivative formulas

An interesting result can be obtained by considering the sum

$$J_{\nu-1}(x)+J_{\nu+1}(x)$$

Using (29) the $x^{\nu-1+2k}$ term of this function will be

$$\begin{aligned} &\left(\frac{x}{2}\right)^{\nu-1}\frac{(-1)^k}{k!(k+\nu-1)!}\left(\frac{x}{2}\right)^{2k} \\ &+\left(\frac{x}{2}\right)^{\nu+1}\frac{(-1)^{k-1}}{(k-1)!(k-1+\nu+1)!}\left(\frac{x}{2}\right)^{2k-2} \\ &=\left(\frac{x}{2}\right)^{\nu-1}\left(\frac{x}{2}\right)^{2k}\frac{(-1)^k}{k!(k+\nu)!}\left[(k+\nu)-k\right] \\ &=\frac{2\nu}{x}\left(\frac{x}{2}\right)^{\nu}\left(\frac{x}{2}\right)^{2k}\frac{(-1)^k}{k!(k+\nu)!} \end{aligned} \tag{62}$$

This is just $2\nu/x$ times a term from the $J_\nu(x)$ series. We therefore have

$$J_{\nu-1}(x)+J_{\nu+1}(x)=\frac{2\nu}{x}J_\nu(x) \tag{63}$$

An almost identical process shows that

$$J_{\nu-1}(x)-J_{\nu+1}(x)=2J_\nu'(x) \tag{64}$$

Using these and the definition of $Y_\nu(x)$ one can show that $Y_\nu(x)$ satisfies the same relations. So, with B_ν representing either J_ν or Y_ν, or indeed any linear combination of these, we have

$$B_{\nu+1}(x)=\frac{2\nu}{x}B_\nu(x)-B_{\nu-1}(x) \tag{65}$$

and

$$B_\nu'(x)=\frac{1}{2}\left[B_{\nu-1}(x)-B_{\nu+1}(x)\right] \tag{66}$$

The recursion (65) is very useful. For example, if are able to calculate just $J_0(x)$ and $J_1(x)$ then we can use (65) to calculate $J_2(x), J_3(x),\ldots$ directly from $J_0(x)$ and $J_1(x)$ without having to use series solutions for all of them. The derivative relation (66) is quite useful because Bessel functions derivatives arise in many applications. This allows us to calculate those directly from the Bessel functions.

Half-integer-order Bessel functions

Starting with

$$\begin{aligned} J_{-1/2}(x) &= \sqrt{\frac{2}{\pi x}}\cos(x) \\ J_{1/2}(x) &= \sqrt{\frac{2}{\pi x}}\sin(x) \end{aligned} \tag{67}$$

we can use (65) to obtain $J_{m+1/2}(x)$ for any integer *m*. For example

$$\begin{aligned} J_{3/2}(x) &= \frac{2(1/2)}{x}J_{1/2}(x)-J_{-1/2}(x) \\ &= \sqrt{\frac{2}{\pi x}}\left[\frac{\sin(x)}{x}-\cos(x)\right] \end{aligned} \tag{68}$$

and so on. From the definition of $Y_\nu(x)$

$$\begin{aligned} Y_{m+1/2}(x) &= \frac{J_{m+1/2}(x)\cos(m+1/2)\pi-J_{-(m+1/2)}(x)}{\sin(m+1/2)\pi} \\ &= (-1)^{m+1}J_{-(m+1/2)}(x) \end{aligned} \tag{69}$$

We see that the half-integer Bessel functions are actually just finite sums of elementary functions. These functions naturally arise in spherical coordinates, and we will call something closely related to them *spherical Bessel functions*.

Wronskian

In Bessel's equation $p(x)=1/x$. Therefore the formula for the Wronskian, equation (23) of Lecture 1c, is

$$W(x)=W(x_0)e^{-\int_{x_0}^{x}\frac{dt}{t}}=W(x_0)\frac{x_0}{x} \tag{70}$$

So the Wronskian behaves as a constant times the inverse of *x*. Since this is true for all *x*, we can consider $x\to 0$ and use the small-argument asymptotic forms

$$\begin{aligned} &J_\nu(x)Y_\nu'(x)-J_\nu'(x)Y_\nu(x) \\ &\qquad\sim \frac{1}{\nu!}\left(\frac{x}{2}\right)^{\nu}\frac{\nu!}{2\pi}\left(\frac{x}{2}\right)^{-\nu-1} \\ &\qquad\qquad + \frac{1}{2(\nu-1)!}\left(\frac{x}{2}\right)^{\nu-1}\frac{(\nu-1)!}{\pi}\left(\frac{x}{2}\right)^{-\nu} \\ &\qquad = \frac{2}{\pi x} \end{aligned} \tag{71}$$

Therefore

$$J_\nu(x)Y_\nu'(x)-J_\nu'(x)Y_\nu(x)=\frac{2}{\pi x} \tag{72}$$

for all *x*. This is useful in some applications.

Hankel functions

The $x\to\infty$ asymptotic forms of the Bessel functions and Euler's formula $\cos\theta+j\sin\theta=e^{j\theta}$ suggest that combinations of the form $J_\nu(x)\pm jY_\nu(x)$ might be useful. We are led to define the *Hankel functions* of the 1st and 2nd kind as

$$\begin{aligned} H_\nu^{(1)}(x)&=J_\nu(x)+jY_\nu(x) \\ H_\nu^{(2)}(x)&=J_\nu(x)-jY_\nu(x) \end{aligned} \tag{73}$$

Their asymptotic forms follow from (61)

$$\begin{aligned} H_\nu^{(1)}(x)&\sim(-j)^{\nu+\frac{1}{2}}\sqrt{\frac{2}{\pi x}}\;e^{jx} \\ H_\nu^{(2)}(x)&\sim\quad j^{\nu+\frac{1}{2}}\sqrt{\frac{2}{\pi x}}\;e^{-jx} \end{aligned} \tag{74}$$

Here we've used

$$e^{-j\left(m+\frac{1}{2}\right)\frac{\pi}{2}}=(-j)^{m+\frac{1}{2}} \tag{75}$$

These are useful, particularly in scattering and radiation problems, because they correspond to *cylindrical waves*. For example

$$H_\nu^{(2)}(\beta\rho)\sim\quad j^{\nu+\frac{1}{2}}\sqrt{\frac{2}{\pi\beta\rho}}\;e^{-j\beta\rho} \tag{76}$$

represents a wave that travels radially outward from the *z* axis toward $\rho=\infty$.

Since the Hankel function are linear combinations of solutions to Bessel's equation they are solutions to Bessel's equation. If ν and *x* are real then $J_\nu(x), Y_\nu(x)$ are real. In this case $H_\nu^{(1)}(x), H_\nu^{(2)}(x)$ are complex conjugates of each other. Also, we have

$$\begin{aligned} H_\nu^{(1)}(x)+H_\nu^{(2)}(x)&=\;\;2J_\nu(x) \\ H_\nu^{(1)}(x)-H_\nu^{(2)}(x)&=j2Y_\nu(x) \end{aligned} \tag{77}$$

so the 1st and 2nd kind of Bessel functions can be written as linear combinations of the Hankel functions. A general solution to Bessel's equation can be formed from a linear combination of any two of the four functions $J_\nu(x), Y_\nu(x), H_\nu^{(1)}(x), H_\nu^{(2)}(x)$. Note that for ν real and non-negative, of these four functions only $J_\nu(x)$ is finite at $x=0$. The other three functions have a singularity there. So if we seek a solution that is finite everywhere, it must be use only $J_\nu(x)$.

References

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