Lecture 4a

Vectors and operators in cylindrical coordinates

Cylindrical coordinates

General cylindrical coordinates u,v,z keep the rectangular z coordinate and specify the x and y coordinates in the form $x=f(u,v)$, $y=g(u,v)$. If the curves $u=\text{const}$ are ellipses or hyperbolas then we would have "elliptical cylindrical coordinates" or "hyperbolic cylindrical coordinates" and so on. The most important case is when $u=\text{const}$ describes a circle in which case we have "circular cylindrical coordinates." These are used so much more than other types of cylindrical coordinates that the term "cylindrical coordinates" implies the circular type by default.

We usually use the coordinate names ρ,ϕ,z and write

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \end{aligned} \tag{1}$$

The curve $\rho=\text{const}$ is a circle of radius ρ which is mapped out as ϕ varies over $0\le\phi<2\pi$ (or any interval of length 2π). The position vector in rectangular coordinates (x,y,z) is

$$\mathbf{r}=(\rho\cos\phi,\rho\sin\phi,z) \tag{2}$$

We have

$$\frac{\partial\mathbf{r}}{\partial\rho}=(\cos\phi,\sin\phi,0) \tag{3}$$

so the corresponding metric coefficient is

$$\left|\frac{\partial\mathbf{r}}{\partial\rho}\right|=h_\rho=1 \tag{4}$$

and the unit vector "in the ρ direction" is

$$\hat{a}_\rho=\hat{a}_x\cos\phi+\hat{a}_y\sin\phi \tag{5}$$

Likewise

$$\frac{\partial\mathbf{r}}{\partial\phi}=(-\rho\sin\phi,\rho\cos\phi,0) \tag{6}$$

so

$$\left|\frac{\partial\mathbf{r}}{\partial\phi}\right|=h_\phi=\rho \tag{7}$$

and

$$\hat{a}_\phi=-\hat{a}_x\sin\phi+\hat{a}_y\cos\phi \tag{8}$$

To summarize, the unit vectors in cylindrical coordinates are

$$\begin{aligned} \hat{a}_\rho &= \hat{a}_x\cos\phi+\hat{a}_y\sin\phi \\ \hat{a}_\phi &= -\hat{a}_x\sin\phi+\hat{a}_y\cos\phi \end{aligned} \tag{9}$$

with $\hat{a}_z$ unchanged from rectangular coordinates.

Differential operators

Identifying $u=\rho, v=\phi, w=z$ and using the metric coefficients $h_\rho=h_z=1, h_\phi=\rho$ we can apply the results of Lecture 1b to obtain the various differential forms in cylindrical coordinates. We have the gradient

$$\nabla f=\hat{a}_\rho\frac{\partial f}{\partial\rho}+\hat{a}_\phi\frac{1}{\rho}\frac{\partial f}{\partial\phi}+\hat{a}_z\frac{\partial f}{\partial z} \tag{10}$$

the divergence

$$\nabla\cdot\mathbf{A}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)+\frac{1}{\rho}\frac{\partial}{\partial\phi}(A_\phi)+\frac{\partial}{\partial z}(A_z) \tag{11}$$

the Laplacian

$$\nabla^2 f=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2} \tag{12}$$

and the curl

$$\begin{aligned}\nabla\times\mathbf{A}= &\ \hat{a}_\rho\left[\frac{1}{\rho}\frac{\partial}{\partial\phi}(A_z)-\frac{\partial}{\partial z}(A_\phi)\right] \\ &+\hat{a}_\phi\left[\frac{\partial}{\partial z}(A_\rho)-\frac{\partial}{\partial\rho}(A_z)\right] \\ &+\frac{\hat{a}_z}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi)-\frac{\partial}{\partial\phi}(A_\rho)\right]\end{aligned} \tag{13}$$

Vector potentials

In rectangular coordinates we saw that we could describe an arbitrary field using only the z components of the magnetic and electric vector potentials. In cylindrical coordinates (of all types) we still have the z coordinate so the same idea applies directly. We need only use the differential forms appropriate for the new coordinate system.

In a source-free region

$$\begin{aligned}\mathbf{H} &= \frac{1}{\mu}\nabla\times\mathbf{A} \\ \mathbf{E} &= \frac{-j}{\omega\epsilon}\nabla\times\mathbf{H}\end{aligned} \tag{14}$$

If $\mathbf{A}=\hat{a}_z A_z$ then using our expression for the curl in cylindrical coordinates we have the general TM^z field

$$\begin{aligned}
H_\rho &= \frac{1}{\mu\rho}\frac{\partial}{\partial\phi}A_z \\
H_\phi &= -\frac{1}{\mu}\frac{\partial}{\partial\rho}A_z \\
H_z &= 0 \\
E_\rho &= -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial\rho}\frac{\partial}{\partial z}A_z \\
E_\phi &= -\frac{j}{\omega\mu\epsilon\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial z}A_z \\
E_z &= -\frac{j}{\omega\mu\epsilon}\left[\beta^2 A_z + \frac{\partial^2}{\partial z^2}A_z\right]
\end{aligned} \tag{15}$$

The E_z expression comes from the fact that (see (10) of Lecture 2c)

$$E_z = \frac{j}{\omega\mu\epsilon}\left[\frac{\partial^2}{\partial x^2}A_z + \frac{\partial^2}{\partial y^2}A_z\right] \tag{16}$$

and

$$\frac{\partial^2}{\partial x^2}A_z + \frac{\partial^2}{\partial y^2}A_z = -\left[\beta^2 A_z + \frac{\partial^2}{\partial z^2}A_z\right] \tag{17}$$

since A_z satisfies the Helmholtz equation. Likewise, in a source-free region

$$\begin{aligned}
\mathbf{E} &= -\frac{1}{\epsilon}\nabla\times\mathbf{F} \\
\mathbf{H} &= \frac{j}{\omega\mu}\nabla\times\mathbf{E}
\end{aligned} \tag{18}$$

so if $\mathbf{F} = \hat{a}_z F_z$ we obtain the general TE^z field.

$$\begin{aligned}
E_\rho &= -\frac{1}{\epsilon\rho}\frac{\partial}{\partial\phi}F_z \\
E_\phi &= \frac{1}{\epsilon}\frac{\partial}{\partial\rho}F_z \\
E_z &= 0 \\
H_\rho &= -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial\rho}\frac{\partial}{\partial z}F_z \\
H_\phi &= -\frac{j}{\omega\mu\epsilon\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial z}F_z \\
H_z &= -\frac{j}{\omega\mu\epsilon}\left[\beta^2 F_z + \frac{\partial^2}{\partial z^2}F_z\right]
\end{aligned} \tag{19}$$

Any field can be represented as a linear combination of TM^z and TE^z components.

Helmholtz equation

As before, the *z* component of either **A** or **F** satisfies the Helmholtz equation

$$\nabla^2 A_z + \beta^2 A_z = 0 \tag{20}$$

To get solutions in cylindrical coordinates we need to use the corresponding expression for the Laplacian. Applying the separation of variables idea, we look for a solution of the form $A_z = f(\rho)\,g(\phi)\,h(z)$. We then have

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\, fgh}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\, fgh}{\partial\phi^2} + \frac{\partial^2\, fgh}{\partial z^2} + \beta^2\, fgh = 0 \tag{21}$$

Diving through by fgh we obtain

$$\frac{1}{\rho}\frac{(\rho f')'}{f} + \frac{1}{\rho^2}\frac{g''}{g} + \frac{h''}{h} + \beta^2 = 0 \tag{22}$$

where $f' = df/d\rho$ and so on. We can isolate the *z* dependence to get

$$\frac{h''}{h} = -\left[\frac{1}{\rho}\frac{(\rho f')'}{f} + \frac{1}{\rho^2}\frac{g''}{g} + \beta^2\right] \tag{23}$$

It follows that both sides must be equal to a constant. We write

$$\frac{h''}{h} = -\beta_z^2 \tag{24}$$

where $-\beta_z^2$ is an arbitrary complex constant. In practice β_z will typically be a positive real number. The general solution for the *z* dependence can be represented by the linear combination

$$h(z) = \begin{Bmatrix} e^{-j\beta_z z} \\ e^{j\beta_z z} \end{Bmatrix} \tag{25}$$

The Helmholtz equation now reads

$$\frac{1}{\rho}\frac{(\rho f')'}{f} + \frac{1}{\rho^2}\frac{g''}{g} + \beta^2 - \beta_z^2 = 0 \tag{26}$$

We can isolate the ϕ dependence to get

$$\frac{g''}{g} = -\rho^2\left[\frac{1}{\rho}\frac{(\rho f')'}{f} + \beta^2 - \beta_z^2\right] \tag{27}$$

We therefore have

$$\frac{g''}{g} = -m^2 \tag{28}$$

We use $-m^2$ instead of $-\beta_\phi^2$ as our separation constant because, as we'll see, most often *m* will end up being an integer. However, nothing keeps *m* from being an arbitrary complex number, so $-m^2$ is a completely general complex constant. The general solution for the ϕ dependence can be represented by the linear combination

$$g(\phi) = \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix} \tag{29}$$

This leaves only the ρ dependence. We have

$$\frac{1}{\rho}\frac{(\rho f')'}{f} - \frac{m^2}{\rho^2} + \beta^2 - \beta_z^2 = 0 \tag{30}$$

Let's call $\beta_\rho^2 = \beta^2 - \beta_z^2$ and multiply through by f to get

$$f''+\frac{1}{\rho}f'+\left[\beta_\rho^2-\frac{m^2}{\rho^2}\right]f=0 \tag{31}$$

When analyzing differential equations it is usually best to put them in a form where all parameters are dimensionless. Dividing through by β_ρ^2 we obtain

$$\frac{1}{\beta_\rho^2}\frac{d^2 f}{d\rho^2}+\frac{1}{\beta_\rho^2\rho}\frac{d f}{d\rho}+\left[1-\frac{m^2}{(\beta_\rho\rho)^2}\right]f=0 \tag{32}$$

Let's define

$$\begin{gathered} x=\beta_\rho\rho \\ y(x)=f(\rho) \end{gathered} \tag{33}$$

Here x and y are simply dimensionless variables and not rectangular coordinates. From the Chain Rule we have

$$\frac{d f}{d\rho}=\frac{d y}{d x}\frac{d x}{d\rho}=\beta_\rho y' \tag{34}$$

and likewise $d^2 f/d\rho^2=\beta_\rho^2 y''$. Applying these we get the standard form of *Bessel's equation*:

$$y''+\frac{1}{x}y'+\left(1-\frac{m^2}{x^2}\right)y=0 \tag{35}$$

Multiplying through by x^2 gives a form that is more convenient for analysis

$$x^2 y''+x y'+\left(x^2-m^2\right)y=0 \tag{36}$$

In the next lecture we will solve this equation in detail. Our solutions will be called the *Bessel functions of the first and second kind of order m*, and will be denoted by $J_m(x), Y_m(x)$. Our solution for the ρ will therefore be represented by the linear combination

$$f(\rho)=\begin{Bmatrix} J_m(\beta_\rho\rho) \\ Y_m(\beta_\rho\rho) \end{Bmatrix} \tag{37}$$

Our general separation of variables solution has the form

$$A(\rho,\phi,z)=\begin{Bmatrix} J_m(\beta_\rho\rho) \\ Y_m(\beta_\rho\rho) \end{Bmatrix}\begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix}\begin{Bmatrix} e^{-j\beta_z z} \\ e^{j\beta_z z} \end{Bmatrix} \tag{38}$$

with the constraint

$$\beta_\rho^2+\beta_z^2=\beta^2 \tag{39}$$