Lecture 3i

Dielectric-Covered Ground Plane

Introduction

For wave guiding structures with PEC (or PMC) surfaces the Poynting vector is non-zero only inside the waveguide cross section (for example, $0\le x\le a, 0\le y\le b$). When waveguides are made with dielectric materials the field can extend outside of the guide, in principle all the way to infinity.

In this lecture we will consider the *dielectric-covered ground plane* waveguide. This structure combines both PEC and dielectric surfaces, as shown in the following figure.

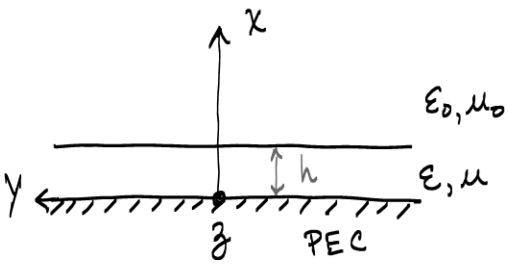

Figure 1:Geometry of the dielectric-covered ground plane problem.

The plane $x=0$ is a PEC surface (the "ground plane"). This has a dielectric coating of thickness *h*. The permeability and permittivity of the dielectric are μ, ϵ. The region $x>h$ is air (essentially free space) with parameters μ_0, ϵ_0. The boundary conditions are

$$\begin{gathered} E_y=E_z=0 \text{ at } x=0 \\ E_y, E_z, H_y, H_z \text{ continuous at } x=h \end{gathered} \quad (1)$$

The subtlety is that the field will be propagating in two different media. While there is a field in the region $x>h$, if this structure is to act as a waveguide the field power density must in some sense be "concentrated" near the dielectric. Consequently we should seek solutions for which the field decays as $x\to\infty$.

TM^z modes

We will investigate TM^z modes that propagate in the *z* direction. Since the boundary conditions at $x=h$ must hold for all y,z, the y and *z* dependence of the fields must be the same for both the $0\le x\le h$ and $x>h$ regions. We will treat the relatively simple case in which there is no *y* dependence, and the *z* dependence of the fields has the form $e^{-j\beta_z z}$. For this case the non-zero field components are

$$\begin{aligned} E_x &= -\frac{j}{\omega\mu\epsilon}\frac{\partial^2}{\partial x\partial z}A_z \\ E_z &= \frac{j}{\omega\mu\epsilon}\frac{\partial^2}{\partial x^2}A_z \\ H_y &= -\frac{1}{\mu}\frac{\partial}{\partial x}A_z \end{aligned} \quad (2)$$

In the region $0\le x\le h$ we can take

$$A_z = A_1\begin{Bmatrix}\cos(\beta_{1x}x)\\ \sin(\beta_{1x}x)\end{Bmatrix}e^{-j\beta_z z} \quad (3)$$

where $\beta_{1x}^2+\beta_z^2=\omega^2\mu\epsilon$. Since we require $E_z=0$ at $x=0$ we need to use the $\sin(\beta_{1x}x)$ factor for A_z. In the region $x>h$ we can take

$$A_z = A_0\begin{Bmatrix}e^{-j\beta_{0x}x}\\ e^{j\beta_{0x}x}\end{Bmatrix}e^{-j\beta_z z} \quad (4)$$

where $\beta_{0x}^2+\beta_z^2=\omega^2\mu_0\epsilon_0$. To get wave guiding, we want this field to decay as $x\to\infty$. If $\beta_{0x}=-j\alpha_{0x}$ then $e^{-j\beta_{0x}x}=e^{-\alpha_{0x}x}$ and $-\alpha_{0x}^2+\beta_z^2=\omega^2\mu_0\epsilon_0$. Therefore, our solution has the form

$$A_z = \begin{cases} A_1\sin(\beta_{1x}x)e^{-j\beta_z z} & 0\le x\le h \\ A_0 e^{-\alpha_{0x}x}e^{-j\beta_z z} & x>h \end{cases} \quad (5)$$

with

$$\begin{aligned} \omega^2\mu_0\epsilon_0 &= \beta_z^2-\alpha_{0x} \\ \omega^2\mu\epsilon &= \beta_z^2+\beta_{1x}^2 \end{aligned} \quad (6)$$

Subtracting the first of these equations from the second gives us the following relation between β_{1x}, α_{0x}

$$\beta_{1x}^2+\alpha_{0x}^2=\omega^2(\mu\epsilon-\mu_0\epsilon_0) \quad (7)$$

We now need to enforce continuity of

$$\begin{aligned} E_z &= \frac{j}{\omega\mu\epsilon}\frac{\partial^2}{\partial x^2}A_z \\ H_y &= -\frac{1}{\mu}\frac{\partial}{\partial x}A_z \end{aligned} \quad (8)$$

at $x=h$. Applied to expression (5) this gives us the following two equations

$$\begin{aligned} \frac{1}{\mu\epsilon}(-\beta_{1x}^2)A_1\sin(\beta_{1x}h) &= \frac{1}{\mu_0\epsilon_0}\alpha_{0x}^2A_0e^{-\alpha_{0x}h} \\ \frac{1}{\mu}\beta_{1x}A_1\cos(\beta_{1x}h) &= \frac{1}{\mu_0}(-\alpha_{0x})A_0e^{-\alpha_{0x}h} \end{aligned} \quad (9)$$

At this point we would like to solve for β_{1x}, α_{0x} (we can worry about A_0, A_1 later). Dividing the first equation by the second gives us the single equation

$$\frac{1}{\epsilon}\beta_{1x}\tan(\beta_{1x}h)=\frac{1}{\epsilon_0}\alpha_{0x} \qquad (10)$$

Together with (7) this gives us two equations in the two unknowns β_{1x},α_{0x}. A change to dimensionless variables is convenient. Let $u=\beta_{1x}h$, $w=\alpha_{0x}h$ and $\epsilon_r=\epsilon/\epsilon_0$. Then the previous equation reads

$$w=\frac{1}{\epsilon_r}u\tan(u) \qquad (11)$$

Multiplying (7) by h^2 we have

$$\beta_{1x}^2 h^2+\alpha_{0x}^2 h^2=\omega^2 h^2(\mu\epsilon-\mu_0\epsilon_0) \qquad (12)$$

This becomes

$$u^2+w^2=V^2 \qquad (13)$$

where

$$V=\omega h\sqrt{\mu\epsilon-\mu_0\epsilon_0} \qquad (14)$$

is called the *normalized frequency*. It is dimensionless and proportional to the frequency. Our two equations can therefore be written

$$\begin{aligned} w&=\frac{1}{\epsilon_r}u\tan(u)\\ w&=\sqrt{V^2-u^2}\end{aligned} \qquad (15)$$

These are represented graphically in the following figure.

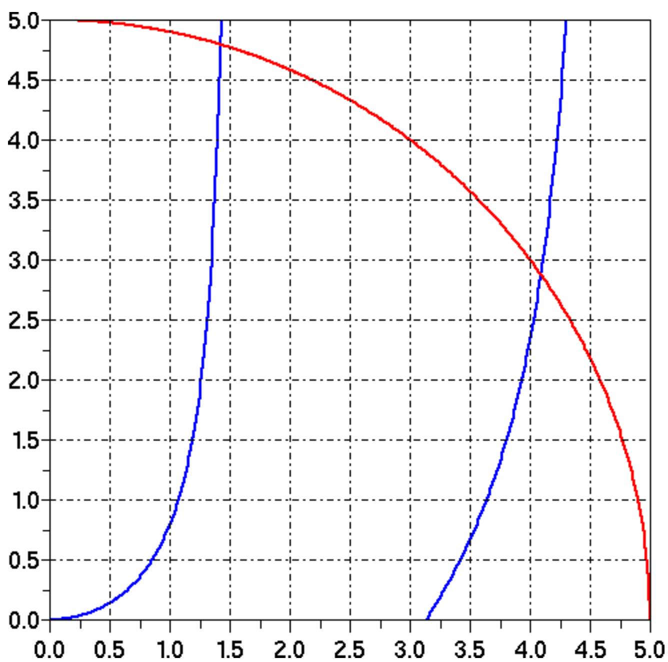

Figure 2: Graphical solution of equations (15) for TM^z modes. Vertical axis is "w" while horizontal axis is "u". In the case plotted $V=5$ and $\epsilon_r=2$. Each intersection gives a solution and corresponds to a particular mode.

The equation $w=\sqrt{V^2-u^2}$ represents a circle of radius *V*. Since $u\tan u$ goes from 0 to infinity as u ranges over $k\pi\le u\le(k+1/2)\pi$ for integer *k*, the equation $w=(1/\epsilon_r)u\tan(u)$ produces a series of curves starting at $u=k\pi, w=0$ and extending to $u=(k+1/2)\pi, w=\infty$. Each intersection of one of these curves with the circle $w=\sqrt{V^2-u^2}$ gives a solution of the system (15). This gives particular *u,w* values which in turn give particular β_{1x},α_{0x} values. From there β_z can be obtained. Finally, A_1 or A_0 can be arbitrarily specified and the other solved for using (9).

As frequency decreases, *V* decreases and the circle $w=\sqrt{V^2-u^2}$ shrinks. If $V<k\pi$ there will be no intersection between the circle and the curve passing through $u=k\pi, w=0$, and that mode will not propagate (it will be below cutoff). The mode cutoff frequencies are therefore given by $\omega h\sqrt{\mu\epsilon-\mu_0\epsilon_0}=k\pi$ or

$$f_k=\frac{k}{2h\sqrt{\mu\epsilon-\mu_0\epsilon_0}} \qquad (16)$$

for $k=0,1,2,\ldots$.

Dominant mode

For $f<1/(2h\sqrt{\mu\epsilon-\mu_0\epsilon_0})$, or $V<\pi$, only one TM^z mode exists, the TM_0^z mode. This mode has no cutoff frequency (that is, the "cutoff frequency" is zero). An analysis of the TE^z modes shows that the lowest TE^z cutoff frequency is $V=\pi/2$ or $f=1/(4h\sqrt{\mu\epsilon-\mu_0\epsilon_0})$. Therefore, if $V<\pi/2$ or $f<1/(4h\sqrt{\mu\epsilon-\mu_0\epsilon_0})$ the waveguide has single-mode operation in the dominant TM_0^z mode.

Suppose we have specific values for ϵ,μ and *h*. Let's trace through the steps required to calculate dominant mode fields. First, given the frequency ω, we calculate the normalized frequency

$$V=\omega h\sqrt{\mu\epsilon-\mu_0\epsilon_0} \qquad (17)$$

Second, solve (15) graphically, or solve

$$\frac{1}{\epsilon_r}u\tan(u)=\sqrt{V^2-u^2} \qquad (18)$$

for *u*. This equation does not have a closed-form solution. However, if *u* is small enough that $\tan u\approx u$ is valid, then it reduces to $u^2=\epsilon_r\sqrt{V^2-u^2}$, or

$$u^4+\epsilon_r^2u^2-(\epsilon_r V)^2=0 \qquad (19)$$

which is a quadratic in u^2 and can be solved analytically to give

$$u=\frac{\epsilon_r}{\sqrt{2}}\left(\sqrt{1+(2V/\epsilon_r)^2}-1\right)^{1/2} \qquad (20)$$

In any case your *u* value fixes

$$\beta_{1x} = \frac{u}{h} \tag{21}$$

Third, $w = \sqrt{V^2 - u^2}$ fixes

$$\alpha_{0x} = \frac{\sqrt{V^2 - u^2}}{h} \tag{22}$$

Fourth, solve for β_z using

$$\beta_z = \sqrt{\omega^2 \mu \epsilon - \beta_{1x}^2} \tag{23}$$

Fifth, chose a value for either A_0 or A_1 and use one of equations (9) to solve for the other. For instance, multiplying the second of these by μh gives

$$A_1 u \cos u = -A_0 \mu_r w e^{-w} \tag{24}$$

You now have the complete solution

$$A_z = \begin{cases} A_1 \sin(\beta_{1x} x) e^{-j\beta_z z} & 0 \le x \le h \\ A_0 e^{-\alpha_{0x} x} e^{-j\beta_z z} & x > h \end{cases} \tag{25}$$

Sixth, and finally, use

$$\begin{aligned} E_x &= -\frac{j}{\omega \mu \epsilon} \frac{\partial^2}{\partial x \partial z} A_z \\ E_z &= \frac{j}{\omega \mu \epsilon} \frac{\partial^2}{\partial x^2} A_z \\ H_y &= -\frac{1}{\mu} \frac{\partial}{\partial x} A_z \end{aligned} \tag{26}$$

to calculate the field components. The field inside the dielectric is

$$\begin{aligned} E_x &= -A_1 \frac{\beta_{1x} \beta_z}{\omega \mu \epsilon} \cos(\beta_{1x} x) e^{-j\beta_z z} \\ E_z &= -j A_1 \frac{\beta_{1x}^2}{\omega \mu \epsilon} \sin(\beta_{1x} x) e^{-j\beta_z z} \\ H_y &= -A_1 \frac{\beta_{1x}}{\mu} \cos(\beta_{1x} x) e^{-j\beta_z z} \end{aligned} \tag{27}$$

Or, calling

$$E_1 = -A_1 \frac{\beta_{1x} \beta_z}{\omega \mu \epsilon} \tag{28}$$

and

$$Z_1 = \frac{\beta_z}{\omega \epsilon} \tag{29}$$

we have

$$\begin{aligned} E_x &= E_1 \cos(\beta_{1x} x) e^{-j\beta_z z} \\ E_z &= j \frac{\beta_{1x}}{\beta_z} E_1 \sin(\beta_{1x} x) e^{-j\beta_z z} \\ H_y &= \frac{E_1}{Z_1} \cos(\beta_{1x} x) e^{-j\beta_z z} \end{aligned} \tag{30}$$

The Poynting vector is

$$\mathbf{P}_1 = \hat{a}_z \frac{1}{2} \frac{|E_1|^2}{Z_1} \cos^2(\beta_{1x} x) \tag{31}$$

Note that P_1 is strongest near the PEC ($x = 0$). The surface current on the PEC is $\mathbf{J}_s = \hat{a}_x \times \mathbf{H}$ or

$$\mathbf{J}_s = \hat{a}_z \frac{E_1}{Z_1} e^{-j\beta_z z} \tag{32}$$

The field outside the dielectric is

$$\begin{aligned} E_x &= A_0 \frac{\alpha_{0x} \beta_z}{\omega \mu_0 \epsilon_0} e^{-\alpha_{0x} x} e^{-j\beta_z z} \\ E_z &= j A_0 \frac{\alpha_{0x}^2}{\omega \mu_0 \epsilon_0} e^{-\alpha_{0x} x} e^{-j\beta_z z} \\ H_y &= A_0 \frac{\alpha_{0x}}{\mu_0} e^{-\alpha_{0x} x} e^{-j\beta_z z} \end{aligned} \tag{33}$$

Calling

$$E_0 = A_0 \frac{\alpha_{0x} \beta_z}{\omega \mu_0 \epsilon_0} \tag{34}$$

and

$$Z_0 = \frac{\beta_z}{\omega \epsilon_0} \tag{35}$$

we have

$$\begin{aligned} E_x &= E_0 e^{-\alpha_{0x} x} e^{-j\beta_z z} \\ E_z &= j E_0 \frac{\alpha_{0x}}{\beta_z} e^{-\alpha_{0x} x} e^{-j\beta_z z} \\ H_y &= \frac{E_0}{Z_0} e^{-\alpha_{0x} x} e^{-j\beta_z z} \end{aligned} \tag{36}$$

The Poynting vector is

$$\mathbf{P}_1 = \hat{a}_z \frac{1}{2} \frac{|E_0|^2}{Z_0} e^{-2\alpha_{0x} x} \tag{37}$$

This decays exponentially in *x* so the energy is localized near the dielectric.

Microstrip

Closely related to the problem we have just considered is the so-called *microstrip* transmission line. This is illustrated below. Microstrip is ubiquitous in RF circuits. At low frequencies it can be treated like a standard PC circuit board structure with a ground plane backing. At high frequencies, however, it must be treated as a waveguide.

The microstrip waveguide is identical to the dielectric-covered ground plane with the addition of a conducting strip of width *w* placed on top of the dielectric. This additional PEC modifies the boundary conditions to require $E_y = E_z = 0$ for $x = h$ and $|y| < w/2$. A rigorous solution to the microstrip problem is a substantial undertaking. One approach is to extend our analysis to include y dependence in the ground-plane modes (both TE and TM) and express the microstrip field as a superposition of the ground-plane modes such that the additional PEC boundary condition is met.

In the homework we considered a rectangular dielectric waveguide covered with PEC on the top and bottom. This can be taken as a simplistic model for microstrip.

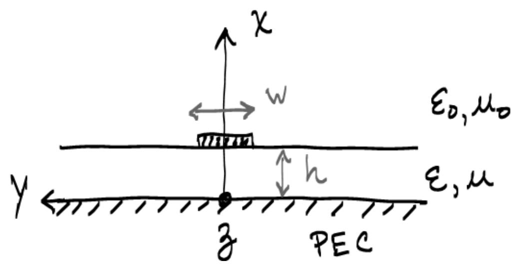

Figure 3: Microstrip transmission line geometry.

References

1. Gowar, J., *Optical Communication Systems*, 2nd Ed., Prentice Hall, 1993, ISBN 0-13-638727-6.