Lecture 3e

Rectangular waveguide

Introduction

So far in rectangular coordinates we have dealt with plane waves propagating in simple and inhomogeneous media. The power density of a plane wave extends over all space. Therefore an ideal plane wave would carry infinite power and as such is not physically realizable. In many applications we need to contain the power of an electromagnetic wave within some specific volume. We can accomplish this using a *waveguide*. Examples of waveguides are coaxial cables, fiber-optics, and TV antenna wires. As we will see, when one confines a wave the possible solutions form a discrete set of *modes*. Each mode typically has a *cutoff frequency* below which the mode cannot propagate, and the propagation constant typically has a non-linear dependence on frequency that leads to the phenomenon of *waveguide dispersion*.

Rectangular waveguide

In this lecture we will consider *rectangular waveguides* which guide waves along the *z* axis while confining their power to a rectangle of dimensions *a* by *b*. This is illustrated in the following figure.

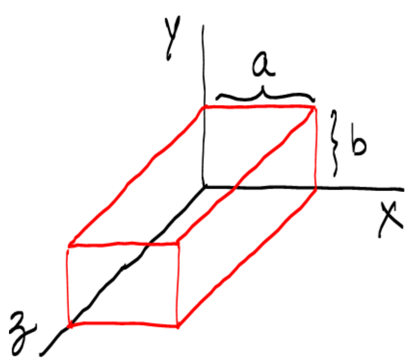

Figure 1: Rectangular waveguide. The guide extends to $\pm\infty$ along the z axis. We take $a \geq b$.

The interior of the waveguide is $0 \leq x \leq a$, $0 \leq y \leq b$ and $-\infty \leq z \leq \infty$. The waveguide surface is typically metallic (ideally PEC), and the interior is typically air-filled (ideally free space). However, we could have PMC or dielectric versions of the same geometry, and the interior could, in principle, be filled with an arbitrary material.

For a PEC waveguide, the boundary conditions – that the tangential electric fields are zero – becomes

$$\begin{array}{ll} E_y = E_z = 0 & x = 0, a \\ E_x = E_z = 0 & y = 0, b \end{array} \tag{1}$$

As we have seen previously, any field can be represented using just the *z* components of the magnetic and electric vector potentials – A_z, F_z. We will begin with the case of a field described by F_z alone. This field will have no E_z component, so we will call it a TE^z mode (the electric field is transverse to the *z* direction).

TE^z Modes

Using our separation of variables results for the Helmholtz equation in rectangular coordinates, we look for solutions of the form

$$F_z = \begin{Bmatrix} \cos(\beta_x x) \\ \sin(\beta_x x) \end{Bmatrix} \begin{Bmatrix} \cos(\beta_y y) \\ \sin(\beta_y y) \end{Bmatrix} e^{-j\beta_z z} \tag{2}$$

We use the $e^{-j\beta_z z}$ factor because we are interested in waves propagating along the *z* axis. As before, the brace notation refers to an arbitrary linear combination of the enclosed terms. Let's consider the boundary condition $E_y = 0$ at $x = 0, a$. We have

$$E_y = \frac{1}{\epsilon} \frac{\partial}{\partial x} F_z \propto \beta_x \begin{Bmatrix} -\sin(\beta_x x) \\ \cos(\beta_x x) \end{Bmatrix} \tag{3}$$

Since we are enforcing the boundary conditions at different values of *x*, we need only consider that factor of F_z that has *x* dependence. To get $E_y = 0$ at $x = 0$ we must use only the $\sin(\beta_x x)$ dependence in E_y, or we must have $\beta_x = 0$. The condition $E_y = 0$ at $x = a$ requires $\beta_x = 0$ or

$$\sin(\beta_x a) = 0 \tag{4}$$

This last condition gives

$$\beta_x = m \frac{\pi}{a} \tag{5}$$

where m is an integer: $m = 0, 1, 2, \ldots$. Since $m = 0$ gives $\beta_x = 0$ we can simply say that the *x* dependence of E_y must be through a factor $\sin(\beta_x x)$ where β_x is one of the values given by (5). Therefore the x dependence of F_z must be through a factor of $\cos(\beta_x x)$. Note that if $m = 0$, so $\beta_x = 0$, then $E_y \equiv 0$ everywhere.

Now consider the boundary conditions $E_x = 0$ at $y = 0, b$. We have

$$E_x = -\frac{1}{\epsilon} \frac{\partial}{\partial y} F_z \propto \beta_y \begin{Bmatrix} -\sin(\beta_y y) \\ \cos(\beta_y y) \end{Bmatrix} \tag{6}$$

This is analogous to the E_y case, so we must have that F_z depends on *y* through a factor of $\cos(\beta_y y)$ with

$$\beta_y = n \frac{\pi}{b} \tag{7}$$

and $n = 0, 1, 2, \ldots$. If $n = 0$ then $E_x \equiv 0$ everywhere. We see that the case $m = n = 0$ is trivial because we would then have

$E_x \equiv E_y \equiv 0$ and there would be no field anywhere.

We have found that solutions exist only for certain discrete values of β_x, β_y and have the form

$$F_z = F_0 \cos(m\pi\, x/a)\cos(n\pi\, x/b) e^{-j\beta_z z} \tag{8}$$

We refer to this as the TE^z_{mn} mode. The value of β_z is fixed by $\beta_x^2+\beta_y^2+\beta_z^2=\omega^2\mu\epsilon$, or

$$\beta_z = \sqrt{\omega^2\mu\epsilon-(m\pi/a)^2-(n\pi/b)^2} \tag{9}$$

Using Equation (24) of Lecture 2c, the electric field is

$$\begin{aligned} E_x &= \frac{F_0}{\epsilon}\beta_y \cos(\beta_x x)\sin(\beta_y y) e^{-j\beta_z z} \\ E_y &= -\frac{F_0}{\epsilon}\beta_x \sin(\beta_x x)\cos(\beta_y y) e^{-j\beta_z z} \\ E_z &= 0 \end{aligned} \tag{10}$$

and the magnetic field is

$$\begin{aligned} H_x &= -\frac{E_y}{Z} \\ H_y &= \frac{E_x}{Z} \\ H_z &= -j\frac{F_0}{\omega\mu\epsilon}(\beta_x^2+\beta_y^2)\cos(\beta_x x)\cos(\beta_y y) e^{-j\beta_z z} \end{aligned} \tag{11}$$

where the *wave impedance* is

$$Z = \frac{\omega\mu}{\beta_z} \tag{12}$$

Dominant mode

As we will see, the most important, or *dominant* mode is the so-called TE^z_{10} mode with $m=1, n=0$. In this case $\beta_x = \pi/a$, $\beta_y = 0$ and

$$\beta_z = \sqrt{\omega^2\mu\epsilon-(\pi/a)^2} \tag{13}$$

The non-zero field components are

$$\begin{aligned} E_y &= -\frac{\pi F_0}{a\epsilon}\sin(\beta_x x) e^{-j\beta_z z} \\ H_x &= \frac{\pi\beta_z F_0}{a\omega\mu\epsilon}\sin(\beta_x x) e^{-j\beta_z z} \\ H_z &= -j\frac{\pi^2 F_0}{a^2\omega\mu\epsilon}\cos(\beta_x x) e^{-j\beta_z z} \end{aligned} \tag{14}$$

It is convenient to call $E_0 = -\pi F_0/(a\epsilon)$. Then

$$\begin{aligned} E_y &= E_0 \sin(\pi x/a) e^{-j\beta_z z} \\ H_x &= -\frac{E_0}{Z}\sin(\pi x/a) e^{-j\beta_z z} \\ H_z &= j\frac{\pi E_0}{\omega\mu a}\cos(\pi x/a) e^{-j\beta_z z} \end{aligned} \tag{15}$$

with

$$\beta_z = \sqrt{\omega^2\mu\epsilon-(\pi/a)^2} \tag{16}$$

and

$$Z = \frac{\omega\mu}{\beta_z} \tag{17}$$

Power flow

The Poynting vector is

$$\begin{aligned} \mathbf{P} &= \frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) \\ &= \frac{1}{2}\mathrm{Re}\big[\hat{a}_z(E_x H_y^* - E_y H_x^*) \\ &\qquad + \hat{a}_x E_y H_z^* - \hat{a}_y E_x H_z^*\big] \end{aligned} \tag{18}$$

If the material filling the waveguide is lossless so that μ, ϵ are real, then the $\hat{a}_x, \hat{a}_y$ terms are purely imaginary due to the *j* factor in H_z. In this case

$$\mathbf{P} = \frac{1}{2}\hat{a}_z(|E_x|^2+|E_y|^2)\,\mathrm{Re}\left[\frac{1}{Z^*}\right] \tag{19}$$

If β_z is real then Z is real and

$$\begin{aligned} \mathbf{P} = \hat{a}_z \frac{\beta_z |F_0|^2}{2\omega\mu\epsilon^2}\big[&\beta_y^2\cos^2(\beta_x x)\sin^2(\beta_y y) \\ &+\beta_x^2\sin^2(\beta_x x)\cos^2(\beta_y y)\big] \end{aligned} \tag{20}$$

If $\beta_z = -j\alpha_z$ is imaginary then $\mathbf{P}=0$. for the TE^z_{10} (assuming β_z is real)

$$\mathbf{P} = \frac{|E_0|^2}{2Z}\sin^2(\pi x/a) \tag{21}$$

The total power carried in the waveguide is

$$W = \int_0^b\int_0^a P_z\, dx\, dy \tag{22}$$

For the TE^z_{10} mode

$$W = \frac{|E_0|^2}{4Z}ab \tag{23}$$

Cutoff

From the condition

$$\beta_z = \sqrt{\omega^2\mu\epsilon-(m\pi/a)^2-(n\pi/b)^2} \tag{24}$$

we see that for real μ, ϵ, if ω falls below a certain value then β_z will become imaginary, and the field will no longer propagate along the *z* axis. Instead it will decay as $e^{-\alpha_z z}$ where $\beta_z = -j\alpha_z$. This is consistent with our result above that $\beta_z = -j\alpha_z$ gives $\mathbf{P}=0$. This frequency is called the *cutoff* *frequency*. The mode propagates only above the cutoff frequency. The cutoff frequency (in Hz) is

$$f_c = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{(m\pi/a)^2+(n\pi/b)^2} \tag{25}$$

Different modes (*m,n* values) have different cutoff frequencies. The lowest cutoff frequency for a TE^z_{mn} mode will correspond to $m=1, n=0$ (since $a\geq b$). This gives

$$f_c = \frac{1}{2a\sqrt{\mu\epsilon}} \tag{26}$$

This is the cutoff frequency of the TE^z_{10} mode. The second lowest cutoff will be for the TE^z_{01} mode and is

$$f_c = \frac{1}{2b\sqrt{\mu\epsilon}} \tag{27}$$

Over the frequency range

$$\frac{1}{2a\sqrt{\mu\epsilon}} < f < \frac{1}{2b\sqrt{\mu\epsilon}} \tag{28}$$

only the TE^z_{10} mode is above cutoff, and we say that the waveguide has *single-mode operation*. For this reason the TE^z_{10} mode is sometimes called the *dominant mode*. In the next lecture we will consider TM^z_{mn} modes using the magnetic vector potential, and we will see that the TE^z_{10} cutoff frequency is lower than that of any TM^z_{mn} mode. Note that as we approach cutoff $\beta_z \to 0$ so $Z\to\infty$.

Most general TE^z field

We have found some particular solutions in the separation of variables form. What does this tell us about an arbitrary TE^z field in the waveguide described by some function $F_z(x,y,z)$? Consider the two-dimensional function $F_z(x,y,0)$. For any value of y we can represent the x dependence of this over $0<x<a$ by a cosine series

$$F_z(x,y,0) = \sum_{m=0}^{\infty} F_m(y)\cos(m\pi x/a) \tag{29}$$

where

$$F_0(y) = \frac{1}{a}\int_0^a F_z(x,y,0)\,dx \tag{30}$$

while for $m\geq 1$

$$F_m(y) = \frac{2}{a}\int_0^a F_z(x,y,0)\cos(m\pi x/a)dx \tag{31}$$

Likewise, the coefficients $F_m(y)$ can be expanded in a cosine series over $0<y<b$:

$$F_m(y) = \sum_{n=0}^{\infty} F_{mn}\cos(n\pi y/b) \tag{32}$$

Substituting this into the previous expression we arrive at

$$F_z(x,y,0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_{mn}\cos(m\pi x/a)\cos(n\pi y/b) \tag{33}$$

This is nothing more than a linear combination of our TE^z_{mn} modes at $z=0$. For other values of z we will have

$$F_z(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_{mn}\cos(m\pi x/a)\cos(n\pi y/b)e^{-j\beta_z z} \tag{34}$$

Therefore, any TE^z field that can exist inside the waveguide can be represented by a superposition of the TE^z_{mn} modes, so we don't need to look for additional solutions.

If we operate at a frequency that gives us single-mode operation, then regardless of the field we start with at $z=0$ we will quickly end up with only the TE^z_{10} mode since it is the only mode that can propagate. All other mode components will decay exponentially along the waveguide.

Ohmic Losses

Our solutions derived above have assumed a PEC waveguide surface. If the surface is a “good conductor” but not perfect, then we can apply the surface resistance idea to calculate ohmic losses. The surface current amplitude on a PEC is equal to the tangential magnetic field, and the power dissipated over a surface is

$$W_{\mathrm{diss}} = \frac{1}{2}R_s\iint |\mathbf{H}_{\tan}|^2\,dS \tag{35}$$

where

$$R_s = \sqrt{\frac{\mu_0}{2\epsilon''}} = \sqrt{\frac{\omega\mu_0}{2\sigma}} \tag{36}$$

For the dominant TE^z_{10} mode the fields are given by (15). Consider a section of the waveguide extending Δz along the z axis. On the surfaces $x=0,a$ only H_z is tangential. Since $\cos^2 0=\cos^2\pi=1$, the power dissipated on these two surfaces is

$$(2)\frac{1}{2}R_s\Delta z\frac{\pi^2|E_0|^2}{(\omega\mu a)^2}\int_0^b dy = R_s b\Delta z\frac{\pi^2|E_0|^2}{(\omega\mu a)^2} \tag{37}$$

On the surfaces $y=0,b$ both H_x, H_z are tangential and

$$|\mathbf{H}_{\tan}|^2 = |E_0|^2\left[\frac{1}{Z^2}\sin^2(\pi x/a)+\frac{\pi^2}{(\omega\mu a)^2}\cos^2(\pi x/a)\right] \tag{38}$$

We integrate this over x from 0 to a. Since

$$\int_0^a \sin^2(\pi x/a)\,dx = \int_0^a \cos^2(\pi x/a)dx = \frac{a}{2} \tag{39}$$

we obtain

$$(2)\frac{1}{2}R_s a\Delta z\frac{|E_0|^2}{2}\left[\frac{1}{Z^2}+\frac{\pi^2}{(\omega\mu a)^2}\right] \tag{40}$$

Putting the contributions of all four walls together we arrive at

$$\frac{dW_{\text{diss}}}{dz}=\frac{aR_s}{2Z^2}|E_0|^2\left[1+\frac{\pi^2 Z^2}{(\omega\mu a)^2}\left(1+\frac{2b}{a}\right)\right] \tag{41}$$

This is the power lost per unit length for the dominant TE_{10}^z mode.

Waveguide dispersion

The dependence of β_z on frequency has important implications. Suppose we excite a waveguide mode with a temporal pulse. For example, consider a Gaussian pulse with width τ and center frequency ω_0

$$a_i(t)=a_0 e^{-\frac{1}{2}(t/\tau)^2}e^{j\omega_0 t} \tag{42}$$

The Fourier representation of this

$$a_i(t)=\int_{-\infty}^{\infty}A_i(\omega)e^{j\omega t}\frac{d\omega}{2\pi} \tag{43}$$

has the spectrum

$$A_i(\omega)=a_0\sqrt{2\pi}\,\tau\, e^{-\frac{1}{2}\tau^2(\omega-\omega_0)^2} \tag{44}$$

We see that the spectral width is $1/\tau$ which is the classic result that a narrow pulse implies a large bandwidth and conversely.

Let's investigate what happens when this pulse travels down a waveguide of length L. A field component with frequency ω will experience a phase change of $\exp(-j\beta_z(\omega)L)$. Therefore, the output pulse will be

$$a_o(t)=\int_{-\infty}^{\infty}A_i(\omega)e^{-j\beta_z(\omega)L}e^{j\omega t}\frac{d\omega}{2\pi} \tag{45}$$

Let's expand β_z around the center frequency

$$\beta_z(\omega)=\beta_{z0}+\beta_{z1}(\omega-\omega_0)+\frac{1}{2}\beta_{z2}(\omega-\omega_0)^2+\cdots \tag{46}$$

with

$$\begin{aligned}\beta_{z0}&=\beta_z(\omega_0)\\ \beta_{z1}&=\frac{d}{d\omega}\beta_z(\omega_0)\\ \beta_{z2}&=\frac{d^2}{d\omega^2}\beta_z(\omega_0)\end{aligned} \tag{47}$$

We need to evaluate

$$\begin{aligned}a_o(t)&=e^{-j\beta_{z0}L}a_0\sqrt{2\pi}\,\tau\\ &\int_{-\infty}^{\infty}e^{-\frac{1}{2}\tau^2(\omega-\omega_0)^2}e^{-j\beta_{z1}(\omega-\omega_0)L}e^{-j\frac{1}{2}\beta_{z2}(\omega-\omega_0)^2L}e^{j\omega t}\frac{d\omega}{2\pi}\end{aligned} \tag{48}$$

Calling $u=\omega-\omega_0$ this becomes

$$\begin{aligned}a_o(t)&=e^{-j\beta_{z0}L}a_0\frac{\tau}{\sqrt{2\pi}}e^{j\omega_0 t}\\ &\int_{-\infty}^{\infty}e^{-\frac{1}{2}\tau^2u^2}e^{-j\beta_{z1}Lu}e^{-j\frac{1}{2}\beta_{z2}Lu^2}e^{jut}du\end{aligned} \tag{49}$$

The integral is

$$\int_{-\infty}^{\infty}e^{-\frac{1}{2}(\tau^2-j\beta_{z2}L)u^2}e^{ju(t-\beta_{z1}L)}du \tag{50}$$

Note that the time variable t has been shifted by an amount $\beta_{z1}L$. This tells us that $u_g=1/\beta_{z1}$ is the velocity with which the pulse travels along the waveguide. This is often called the *group velocity*. In addition, the pulse will be broadened. Using

$$\int_{-\infty}^{\infty}e^{-\frac{1}{2}k^2u^2}e^{jut}du=\frac{\sqrt{2\pi}}{k}e^{-\frac{1}{2}\frac{t^2}{k^2}} \tag{51}$$

where $k^2=\tau^2-j\beta_{z2}L$ and

$$\begin{aligned}\frac{1}{\tau^2-j\beta_{z2}L}&=\frac{\tau^2+j\beta_{z2}L}{\tau^4+(\beta_{z2}L)^2}\\ &=\frac{1}{\tau^2+(\beta_{z2}L/\tau)^2}+\frac{j\beta_{z2}L}{\tau^4+(\beta_{z2}L)^2}\end{aligned} \tag{52}$$

we find that the amplitude of the output pulse varies as

$$|a_o(t)|\propto e^{-\frac{1}{2}\frac{(t-\beta_{z1}L)^2}{\tau^2+(\beta_{z2}L/\tau)^2}} \tag{53}$$

The width goes from τ to

$$\tau_o=\sqrt{\tau^2+(\beta_{z2}L/\tau)^2} \tag{54}$$

This is the phenomenon of *waveguide dispersion* in which a pulse widens as it travels down the waveguide. The narrower the original pulse (the small the value of τ) the greater the dispersion. Minimizing $\tau_o^2=\tau^2+(\beta_{z2}L/\tau)^2$ with respect to τ we find

$$\tau_{o,min}=\sqrt{2|\beta_{z2}|L} \tag{55}$$

for the minimum possible output pulse width. This corresponds to an input pulse width $\tau=\sqrt{|\beta_{z2}|L}$. We see that the 2nd derivative (or the curvature) of the curve $\beta_z(\omega)$ limits the pulse widths that can be sent down the waveguide.

Note that for a plane wave in a simple medium with $\beta_z=\omega\sqrt{\mu\epsilon}$, the 2nd derivative of $\beta_z(\omega)$ is zero, so dispersion does not occur. However, if $\epsilon=\epsilon(\omega)$ then the 2nd derivative of $\beta_z(\omega)$ will be non-zero and dispersion will occur. This is called *material dispersion* since it arises from the properties of the material rather than from the geometry of a waveguide. In fiber-optics, where dielectric waveguides are employed, one typically has both material and waveguide dispersion to contend with.

TM^z modes

In the previous lecture we solved for the TE^z modes in a rectangular waveguide. We now consider TM^z modes for which the magnetic vector potential has the form

$$A_z = \begin{Bmatrix} \cos(\beta_x x) \\ \sin(\beta_x x) \end{Bmatrix} \begin{Bmatrix} \cos(\beta_y y) \\ \sin(\beta_y y) \end{Bmatrix} e^{-j\beta_z z} \tag{56}$$

The boundary conditions for a PEC waveguide are as before

$$\begin{aligned} E_y = E_z = 0 \quad x = 0, a \\ E_x = E_z = 0 \quad y = 0, b \end{aligned} \tag{57}$$

The electric field is given by (see Lecture 2c)

$$\begin{aligned} E_x &= -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial x}\frac{\partial}{\partial z}A_z \\ E_y &= -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial y}\frac{\partial}{\partial z}A_z \\ E_z &= \frac{j}{\omega\mu\epsilon}\left[\frac{\partial^2}{\partial x^2}A_z + \frac{\partial^2}{\partial y^2}A_z\right] \end{aligned} \tag{58}$$

Since E_x must vanish at $y=0$ and it does not involve a y derivative of A_z, we must use the $\sin(\beta_y y)$ term in our A_z solution. Likewise, E_y must vanish at $x=0$ and it does not involve an x derivative of A_z, so we must use the $\sin(\beta_x x)$ term. Our solution will therefore have the form

$$A_z = A_0 \sin(\beta_x x)\sin(\beta_y y) e^{-j\beta_z z} \tag{59}$$

The vanishing of E_y at $x=a$ and the vanishing of E_x at $y=b$ require

$$\begin{aligned} \beta_x &= m\frac{\pi}{a} \quad , \quad m = 1,2,\ldots \\ \beta_y &= n\frac{\pi}{b} \quad , \quad n = 1,2,\ldots \end{aligned} \tag{60}$$

Note that neither m or n can be zero since that would lead to $A_z \equiv 0$ and there would be no field. The value of β_z is fixed by the requirement $\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega\mu\epsilon$ to be

$$\beta_z = \sqrt{\omega^2\mu\epsilon - (m\pi/a)^2 - (n\pi/b)^2} \tag{61}$$

The cutoff frequency is therefore

$$f_c = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{(m/a)^2 + (n/b)^2} \tag{62}$$

The lowest cutoff frequency is for the TM_{11}^z mode. Since there is no TM_{10}^z mode we see that the TE_{10}^z mode is indeed the dominant mode in the waveguide. The TE_{01}^z and TE_{20}^z cutoffs will also be lower than that of the TM_{11}^z mode (provided $a > b$). Therefore, the waveguide has single-mode operation over the frequency range

$$\frac{1}{2a\sqrt{\mu\epsilon}} < f < \frac{1}{2a\sqrt{\mu\epsilon}}\min(2, a/b) \tag{63}$$

The upper limit is at least a factor of 2 times the lower limit. We say that the waveguide operates single mode over at least one "octave."

For the TM_{mn}^z mode the electric field is

$$\begin{aligned} E_x &= \frac{-A_0\beta_z}{\omega\mu\epsilon}\beta_x \cos(\beta_x x)\sin(\beta_y y) e^{-j\beta_z z} \\ E_y &= \frac{-A_0\beta_z}{\omega\mu\epsilon}\beta_y \sin(\beta_x x)\cos(\beta_y y) e^{-j\beta_z z} \\ E_z &= \frac{-jA_0}{\omega\mu\epsilon}(\beta_x^2 + \beta_y^2)\sin(\beta_x x)\sin(\beta_y y) e^{-j\beta_z z} \end{aligned} \tag{64}$$

and the magnetic field is

$$\begin{aligned} H_x &= \frac{1}{\mu}\frac{\partial}{\partial y}A_z = \frac{A_0\beta_y}{\mu}\sin(\beta_x x)\cos(\beta_y y) \\ H_y &= -\frac{1}{\mu}\frac{\partial}{\partial x}A_z = \frac{-A_0\beta_x}{\mu}\cos(\beta_x x)\sin(\beta_y y) \\ H_z &= 0 \end{aligned} \tag{65}$$

or

$$\begin{aligned} H_x &= -\frac{E_y}{Z} \\ H_y &= \frac{E_x}{Z} \\ H_z &= 0 \end{aligned} \tag{66}$$

where the wave impedance is

$$Z = \frac{\beta_z}{\omega\epsilon} \tag{67}$$

Rectangular Resonators

Consider the situation in the following illustration.

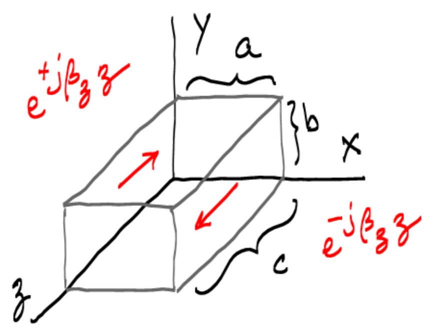

Figure 1: Rectangular resonator.

Here we've cut a length c of the waveguide and capped both ends with PEC plates. We will no longer be able to have power propagating only along the z axis. Instead, we will need waves propagating in both the $+z$ and $-z$ directions. Together these should form a "standing wave" (non-propagating) field of the type we have in the x and y dimensions. Let's consider TE^z modes. We take

$$F_z = \cos(\beta_x x)\cos(\beta_y y)\left(F_1 e^{-j\beta_z z} + F_2 e^{j\beta_z z}\right) \quad (68)$$

This represents a superposition of waves traveling in the the $+z$ and $-z$ directions with amplitudes F_1, F_2. The electric field will have $E_x = -(1/\epsilon)\partial F_z/\partial y$, $E_y = (1/\epsilon)\partial F_z/\partial x$ or

$$\begin{aligned} E_x &= \frac{1}{\epsilon}\beta_y \cos(\beta_x x)\sin(\beta_y y)\left[F_1 e^{-j\beta_z z} + F_2 e^{j\beta_z z}\right] \\ E_y &= -\frac{1}{\epsilon}\beta_x \sin(\beta_x x)\cos(\beta_y y)\left[F_1 e^{-j\beta_z z} + F_2 e^{j\beta_z z}\right] \end{aligned} \quad (69)$$

Both of these components are tangential at $z=0$ and must therefore be zero. This gives the condition $F_1 + F_2 = 0$. Therefore

$$F_1 e^{-j\beta_z z} + F_2 e^{j\beta_z z} = F_1\left(e^{-j\beta_z z} - e^{j\beta_z z}\right) = F_0 \sin(\beta_z z) \quad (70)$$

and

$$F_z = F_0 \cos(\beta_x x)\cos(\beta_y y)\sin(\beta_z z) \quad (71)$$

so

$$\begin{aligned} E_x &= \frac{F_0}{\epsilon}\beta_y \cos(\beta_x x)\sin(\beta_y y)\sin(\beta_z z) \\ E_y &= -\frac{F_0}{\epsilon}\beta_x \sin(\beta_x x)\cos(\beta_y y)\sin(\beta_z z) \end{aligned} \quad (72)$$

Now, $E_x = E_y = 0$ at $z=c$ requires $\sin(\beta_z c) = 0$ or

$$\beta_z = p\frac{\pi}{c} \quad (73)$$

where $p = 1,2,\ldots$. Note that $p=0$ would give $\mathbf{E} \equiv 0$. Taken together with the conditions on TE^z_{mn} waveguide modes we have

$$\begin{aligned} \beta_x &= m\frac{\pi}{a} \quad m = 0,1,\ldots \\ \beta_y &= n\frac{\pi}{b} \quad n = 0,1,\ldots \\ \beta_z &= p\frac{\pi}{c} \quad p = 1,2,\ldots \end{aligned} \quad (74)$$

with the additional constant that both *m* and *n* cannot be zero. Therefore

$$\pi^2\left((m/a)^2 + (n/b)^2 + (p/c)^2\right) = \omega^2 \mu \epsilon \quad (75)$$

so that

$$f_r = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{(m/a)^2 + (n/b)^2 + (p/c)^2} \quad (76)$$

is the *resonant frequency* of the TE^z_{mnp} mode. The lowest resonant frequency will be that of the TE^z_{101} mode (provided $a > b$).

TM^z modes

From our analysis above it's clear that for TM^z resonator modes the magnetic vector potential will have the form

$$A_z = A_0 \sin(\beta_x x)\sin(\beta_y y)\begin{Bmatrix}\cos(\beta_z z) \\ \sin(\beta_z z)\end{Bmatrix} \quad (77)$$

Since

$$\begin{aligned} E_x &= -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial x}\frac{\partial}{\partial z}A_z \\ E_y &= -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial y}\frac{\partial}{\partial z}A_z \end{aligned} \quad (78)$$

we must use the $\cos(\beta_z z)$ factor in our A_z expression so that E_x, E_y are proportional to $\sin(\beta_z z)$ and hence vanish at $z=0$. We have

$$A_z = A_0 \sin(\beta_x x)\sin(\beta_y y)\cos(\beta_z z) \quad (79)$$

The boundary condition $E_x = E_y = 0$ at $z=c$ requires

$$\beta_z = p\frac{\pi}{c} \quad (80)$$

where $p = 0,1,\ldots$. Note that $p=0$ is acceptable. Although this gives $E_x = E_y = 0$ everywhere, $E_z \neq 0$ so there will still be a field. The conditions for a TM^z_{mnp} resonator mode are

$$\begin{aligned} \beta_x &= m\frac{\pi}{a} \quad , \quad m = 1,2,\ldots \\ \beta_y &= n\frac{\pi}{b} \quad , \quad n = 1,2,\ldots \\ \beta_z &= p\frac{\pi}{c} \quad , \quad p = 0,1,\ldots \end{aligned} \quad (81)$$

and the resonant frequency is

$$f_r = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{(m/a)^2 + (n/b)^2 + (p/c)^2} \quad (82)$$

The minimum resonant frequency will be for the TM^z_{110} mode. If $c > b$ then the TE^z_{101} frequency will be lower than the TM^z_{110} frequency. Otherwise the TM^z_{110} frequency will be the lowest.