Lecture 3e

Oblique incidence and wave impedance

Introduction

We now wish to consider the case of an arbitrary plane wave incident on a planar interface or, more generally, a layered medium. The results are similar to the normal incidence case but some additional bookkeeping is required as the reflection and transmission coefficients are polarization dependent.

H polarization

Consider a planar interface at $z=0$ between two simple media with

$$\mu,\epsilon=\begin{cases}\mu_1,\epsilon_1 & z<0\\ \mu_2,\epsilon_2 & z\geq 0\end{cases} \tag{1}$$

In general case μ,ϵ may be either real (lossless) or complex (lossy).

A plane wave with propagation constants β_x,β_y,β_z and arbitrary polarization can be represented using just the *z* components of the magnetic and electric vector potentials

$$\begin{aligned} A_z&=A_0\,e^{-j\beta_x x}\,e^{-j\beta_y y}\,e^{-j\beta_z z}\\ F_z&=F_0\,e^{-j\beta_x x}\,e^{-j\beta_y y}\,e^{-j\beta_z z}\end{aligned} \tag{2}$$

If only A_z is used then the field has no H_z component and **H** is parallel to the interface (the *x*-*y* plane). We will call this the *H polarization* case.

Since our field will have *x* and *y* dependence we will need to make sure that the boundary conditions (tangential **E** and **H** fields are continuous at the interface) hold at all *x* and *y* values. Toward this end, consider an equation of the form

$$e^{-j\beta_{ix}x}+\rho\,e^{-j\beta_{rx}x}=\tau\,e^{-j\beta_{tx}x} \tag{3}$$

For this to hold at all *x*, we must have $\beta_{ix}=\beta_{rx}=\beta_{tx}$ and $1+\rho=\tau$. To see that this is so, consider that if two functions are equal then all their derivatives are equal. Therefore

$$\beta_{ix}^n\,e^{-j\beta_{ix}x}+\rho\,\beta_{rx}^n\,e^{-j\beta_{rx}x}=\tau\,\beta_{tx}^n\,e^{-j\beta_{tx}x} \tag{4}$$

for all $n\geq 0$. This can only be true if $\beta_{ix}=\beta_{rx}=\beta_{tx}$. It follows that what the incident, the reflected and the transmitted fields must have the same *x* and *y* dependence at the interface $z=0$. Therefore, we will represent the incident, reflected and transmitted fields by

$$\begin{aligned} A_{iz}&=A_{i0}\,e^{-j\beta_x x}\,e^{-j\beta_y y}\,e^{-j\beta_{1z} z}\\ A_{rz}&=A_{r0}\,e^{-j\beta_x x}\,e^{-j\beta_y y}\,e^{\,j\beta_{1z} z}\\ A_{tz}&=A_{t0}\,e^{-j\beta_x x}\,e^{-j\beta_y y}\,e^{-j\beta_{2z} z}\end{aligned} \tag{5}$$

so that the *x* and *y* behavior of the three plane waves are identical. The *z* propagation constants follow from the requirement

$$\begin{aligned} \beta_x^2+\beta_y^2+\beta_{1z}^2&=\omega^2\mu_1\epsilon_1\\ \beta_x^2+\beta_y^2+\beta_{2z}^2&=\omega^2\mu_2\epsilon_2\end{aligned} \tag{6}$$

We use $e^{j\beta_{1z}z}$ behavior for the reflected wave to correspond to a wave propagating toward $z\rightarrow-\infty$. The constants A_{i0} and β_x,β_y are fixed by specifying the incident plane wave. We now need to solve for A_{r0},A_{t0}.

The tangential (*x* and *y*) components of a field are derived from A_z using

$$\begin{aligned} H_x&=\frac{1}{\mu}\frac{\partial}{\partial y}A_z\\ H_y&=-\frac{1}{\mu}\frac{\partial}{\partial x}A_z\\ E_x&=-\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial x}\frac{\partial}{\partial z}A_z\\ E_y&=-\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial y}\frac{\partial}{\partial z}A_z\end{aligned} \tag{7}$$

In the normal incidence case we saw that the intrinsic impedance $\eta=\sqrt{\mu/\epsilon}$, which is the ratio of the electric to magnetic field amplitudes, was important. In the present case we will need to consider the ratio of the electric to magnetic tangential field components. For example

$$\frac{E_x}{H_y}=\frac{-\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial x}\frac{\partial}{\partial z}A_z}{-\frac{1}{\mu}\frac{\partial}{\partial x}A_z}=\frac{\beta_z}{\omega\epsilon} \tag{8}$$

This leads us to define the *wave impedance* for H polarization

$$\boxed{Z=\frac{\beta_z}{\omega\epsilon}} \tag{9}$$

Note that when $\beta_z=\beta=\omega\sqrt{\mu\epsilon}$ then $Z=\eta$. In general, however, *Z* depends on the permittivity, permeability and on the direction of propagation of the plane wave. It is therefore not as physically fundamental as η, but is very useful mathematically.

From (7) we see that if

$$\frac{1}{\mu_1}(A_{i0}+A_{r0})=\frac{1}{\mu_2}A_{t0} \tag{10}$$

at $z=0$ then H_x,H_y will be continuous across the interface, because in each case $\partial/\partial x$ or $\partial/\partial y$ will introduce the same constant $-j\beta_x$ or $-j\beta_y$ for both media. To get the E_x,E_y components to be continuous, we require that

$$\frac{j}{\omega\mu_1\epsilon_1}(-j\beta_{1z}A_{i0}+j\beta_{1z}A_{r0})=\frac{j}{\omega\mu_2\epsilon_2}(-j\beta_{2z}A_{t0}) \tag{11}$$

Solving for A_{t0} in both of these equations and equating those expressions gives us

$$A_{t0}=\frac{\mu_2}{\mu_1}(A_{i0}+A_{r0})$$
$$=\frac{\mu_2}{\mu_1}\frac{\epsilon_2}{\epsilon_1}\frac{\beta_{1z}}{\beta_{2z}}(A_{i0}-A_{r0}) \tag{12}$$

We now have a single equation in A_{i0}, A_{r0}. The μ factors cancel, and multiplying by ω/ω we can write

$$A_{i0}+A_{r0}=\frac{Z_1}{Z_2}(A_{i0}-A_{r0}) \tag{13}$$

The solution is

$$A_{r0}=A_{i0}\frac{Z_1-Z_2}{Z_1+Z_2} \tag{14}$$

The transmitted amplitude is

$$A_{t0}=A_{i0}\frac{\mu_2}{\mu_1}\frac{2Z_1}{Z_1+Z_2} \tag{15}$$

We will define the reflection coefficient as the ratio of the tangential components of the transmitted and incident electric fields. We have

$$\rho=\frac{E_{rx}}{E_{ix}}=\frac{E_{ry}}{E_{iy}}=-\frac{A_{r0}}{A_{i0}} \tag{16}$$

The minus sign comes from the $e^{-j\beta_{1z}z}$ and $e^{j\beta_{1z}z}$ z dependences of the incident and reflected fields and the $\partial/\partial z$ operation involved in obtaining E_x from A_z. The transmission coefficient is likewise defined as

$$\tau=\frac{E_{tx}}{E_{ix}}=\frac{E_{ty}}{E_{iy}}=\frac{\beta_{2z}/(\mu_2\epsilon_2)}{\beta_{1z}/(\mu_1\epsilon_1)}\frac{A_{t0}}{A_{i0}}=\frac{\mu_1}{\mu_2}\frac{Z_2}{Z_1}\frac{A_{t0}}{A_{i0}} \tag{17}$$

From our solution for A_{r0}, A_{t0} we have immediately

$$\boxed{\begin{aligned}\rho&=\frac{Z_2-Z_1}{Z_2+Z_1}\\ \tau&=\frac{2Z_2}{Z_2+Z_1}\end{aligned}} \tag{18}$$

These results are identical to the normal incidence case except that we need to use the wave impedance (9) in place of the intrinsic impedance.

E polarization

We will now use the electric vector potential to represent the fields:

$$\begin{aligned}F_{iz}&=F_{i0}e^{-j\beta_x x}e^{-j\beta_y y}e^{-j\beta_{1z}z}\\ F_{rz}&=F_{r0}e^{-j\beta_x x}e^{-j\beta_y y}e^{j\beta_{1z}z}\\ F_{tz}&=F_{0}e^{-j\beta_x x}e^{-j\beta_y y}e^{-j\beta_{2z}z}\end{aligned} \tag{19}$$

These fields will have no E_z component. Since **E** is parallel to the interface we will refer to this as the *E polarization* case. The tangential components of the field are given by

$$\begin{aligned}E_x&=-\frac{1}{\epsilon}\frac{\partial}{\partial y}F_z\\ E_y&=\frac{1}{\epsilon}\frac{\partial}{\partial x}F_z\\ H_x&=-\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial x}\frac{\partial}{\partial z}F_z\\ H_y&=-\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial y}\frac{\partial}{\partial z}F_z\end{aligned} \tag{20}$$

The wave impedance is

$$\frac{E_x}{H_y}=\frac{-\frac{1}{\epsilon}\frac{\partial}{\partial y}F_z}{-\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial y}\frac{\partial}{\partial z}F_z}=\frac{\omega\mu}{\beta_z} \tag{21}$$

or

$$\boxed{Z=\frac{\omega\mu}{\beta_z}} \tag{22}$$

for E polarization. When $\beta_z=\beta=\omega\sqrt{\mu\epsilon}$ then $Z=\eta$. Note that in general the wave impedance is different for the H (9) and E (22) polarizations and can be complex.

From (20) we see that if

$$\frac{1}{\epsilon_1}(F_{i0}+F_{r0})=\frac{1}{\epsilon_2}F_{t0} \tag{23}$$

at $z=0$ then E_x, E_y will be continuous across the interface. To get the H_x, H_y components to be continuous, we require that

$$\frac{j}{\omega\mu_1\epsilon_1}(-j\beta_{1z}F_{i0}+j\beta_{1z}F_{r0})=\frac{j}{\omega\mu_2\epsilon_2}(-j\beta_{2z}F_{t0}) \tag{24}$$

Solving for F_{t0} in both of these equations and equating those expressions gives us

$$\begin{aligned}F_{t0}&=\frac{\epsilon_2}{\epsilon_1}(F_{i0}+F_{r0})\\ &=\frac{\mu_2}{\mu_1}\frac{\epsilon_2}{\epsilon_1}\frac{\beta_{1z}}{\beta_{2z}}(F_{i0}-F_{r0})\end{aligned} \tag{25}$$

or

$$F_{i0}+F_{r0}=\frac{Z_2}{Z_1}(F_{i0}-F_{r0}) \tag{26}$$

The solution is

$$F_{r0}=F_{i0}\frac{Z_2-Z_1}{Z_2+Z_1} \tag{27}$$

Then

$$F_{t0}=F_{i0}\frac{\epsilon_2}{\epsilon_1}\frac{2Z_2}{Z_2+Z_1} \tag{28}$$

The reflection coefficient is

$$\rho=\frac{E_{rx}}{E_{ix}}=\frac{E_{ry}}{E_{iy}}=\frac{F_{r0}}{F_{i0}} \tag{29}$$

and the transmission coefficient is

$$\tau = \frac{E_{tx}}{E_{ix}} = \frac{E_{ty}}{E_{iy}} = \frac{1/\epsilon_2}{1/\epsilon_1}\frac{F_{t0}}{F_{i0}} \quad (30)$$

so

$$\boxed{\begin{aligned}\rho &= \frac{Z_2 - Z_1}{Z_2 + Z_1} \\ \tau &= \frac{2Z_2}{Z_2 + Z_1}\end{aligned}} \quad (31)$$

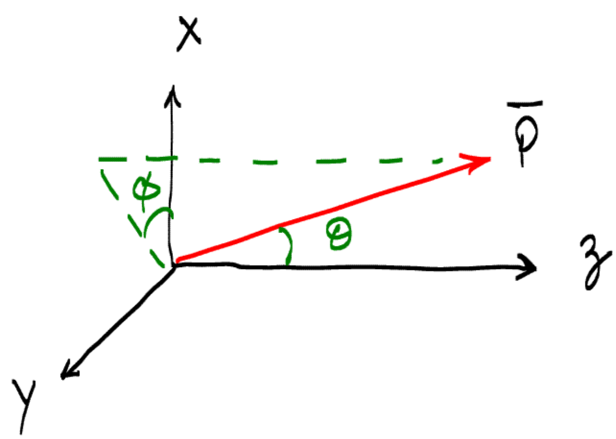
The expressions for ρ, τ are identical in the two cases, but keep in mind that the impedance depends on the particular polarization.

Lossless media

Up until now we have considered the general case where μ, ϵ can be complex in either or both media. If μ, ϵ are real, then $\beta = \omega\sqrt{\mu\epsilon}$ is real. We will write

$$\begin{aligned}\beta_x &= \beta \sin(\theta)\cos(\phi) \\ \beta_y &= \beta \sin(\theta)\sin(\phi) \\ \beta_z &= \beta \cos(\theta)\end{aligned} \quad (32)$$

as the most general set of $\beta_x, \beta_y, \beta_z$ with $\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2$. As illustrated below, the angles θ, ϕ represent the direction of propagation (the Poynting vector) in spherical coordinates.



The angle between **P** and the *z* axis is θ while the projection of **P** into the *x*-*y* plane makes an angle of ϕ with the *x* axis.

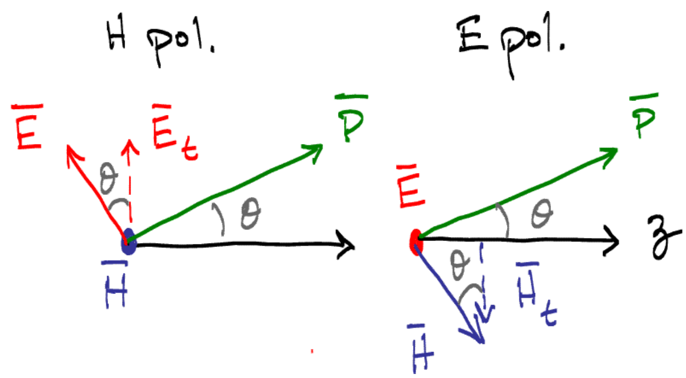
For H polarization the wave impedance (9) is

$$Z = \frac{\beta_z}{\omega\epsilon} = \frac{\omega\sqrt{\mu\epsilon}\cos(\theta)}{\omega\epsilon} = \eta\cos(\theta) \quad (33)$$

Since $|\cos(\theta)| \le 1$ we have $Z \le \eta$. For the E polarization case the wave impedance (22) is

$$Z = \frac{\omega\mu}{\beta_z} = \frac{\omega\mu}{\omega\sqrt{\mu\epsilon}\cos(\theta)} = \eta/\cos(\theta) \quad (34)$$

and $Z \ge \eta$. The geometric interpretation of this is shown in the following figure.



For H polarization the tangential component of E involves a $\cos(\theta)$ projection factor while for E polarization the tangential component of H contains this factor. With *Z* being the ratio of E to H tangential components, we get a $\cos(\theta)$ factor for H polarization and a $1/\cos(\theta)$ factor for E polarization.

Snell's law

If medium 1 has $\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$ and angles θ_1, ϕ_1 and medium 2 has $\beta_2 = \omega\sqrt{\mu_2\epsilon_2}$ and angles θ_2, ϕ_2, then the continuity of β_x, β_y at the boundary requires

$$\begin{aligned}\beta_x &= \beta_1 \sin(\theta_1)\cos(\phi_1) = \beta_2 \sin(\theta_2)\cos(\phi_2) \\ \beta_y &= \beta_1 \sin(\theta_1)\sin(\phi_1) = \beta_2 \sin(\theta_2)\sin(\phi_2)\end{aligned} \quad (35)$$

Taking the ratio of these equations gives $\tan\phi_1 = \tan\phi_2$ or $\phi_2 = \phi_1$. We then have

$$\beta_1 \sin(\theta_1) = \beta_2 \sin(\theta_2) \quad (36)$$

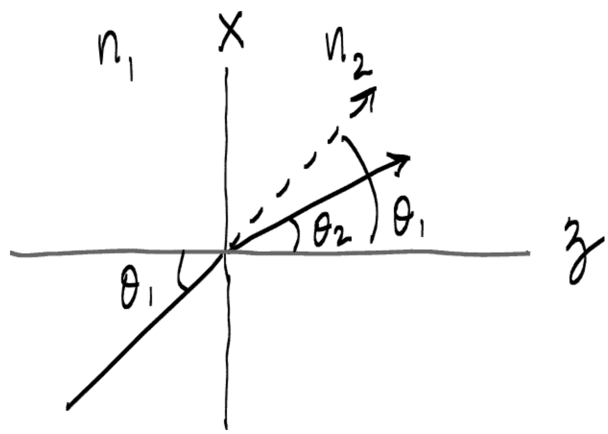
If we divide both sides by $\omega\sqrt{\mu_0\epsilon_0}$ and define the *index of refraction* as

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \sqrt{\mu_r\epsilon_r} \quad (37)$$

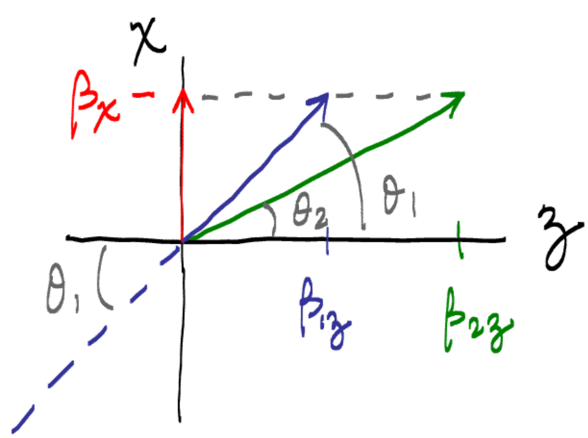
then we obtain *Snell's law*

$$\boxed{n_1 \sin(\theta_1) = n_2 \sin(\theta_2)} \quad (38)$$

This is illustrated below for the case $\phi_2 = \phi_1 = 0$ which gives $\beta_y = 0$ so that the incident and transmitted Poynting vectors all lie in the *x*-*z* plane.



A graphical derivation of Snell's law is given in the following figure. Again, we take $\phi_2=\phi_1=0$ so that $\beta_y=0$. The x projection of each propagation vector (β_x) is the same. Therefore, the differing vector lengths $\beta=\omega\sqrt{\mu\epsilon}$ necessitate different z projections and corresponding different propagation angles $\theta_2=\theta_1$.



Critical angle

Interesting special cases of Snell's law might arise where the incident power is either completely reflected or completely transmitted. With β_x,β_y fixed, β_z is determined by

$$\beta_z=\pm\sqrt{\omega^2\mu\epsilon-\beta_x^2-\beta_y^2} \tag{39}$$

If

$$\omega^2\mu_2\epsilon_2<\beta_x^2+\beta_y^2<\omega^2\mu_1\epsilon_1 \tag{40}$$

then β_z will be real in medium 1 but imaginary in medium 2. That is, in medium 2 we will have $\beta_z=-j\alpha_z$, and the z dependence of the field is exponentially decaying, $e^{-\alpha_z z}$.

In this case, for either E or H polarization, the impedance Z_1 will be real while Z_2 will be imaginary. Therefore $|Z_2-Z_1|=|Z_2+Z_1|$ and $|\rho|^2=1$, and all incident power is reflected. This is sometimes called *total internal reflection*. It is the principle on which fiber optic waveguides are based.

The condition on the angle θ_1 for total internal reflection is

$$\beta_x^2+\beta_y^2=\omega^2\mu_1\epsilon_1\sin^2(\theta_1)\ge\omega^2\mu_2\epsilon_2 \tag{41}$$

or $\theta_1\ge\theta_c$ where the *critical angle* is

$$\theta_c=\sin^{-1}\sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}}=\sin^{-1}\left(\frac{n_2}{n_1}\right) \tag{42}$$

We can also see this from Snell's law. When $\theta_1=\theta_c$ then Snell's law requires $\theta_2=\pi$ and the transmitted field travels parallel to the interface. Therefore, no power is propagated in the z direction, and all incident power must be reflected. When $\theta_1>\theta_c$ Snell's law would require $\sin\theta_2>1$ and there is no real solution for θ_2 . In this case, formally, θ_2 would become complex. For example, if $\theta_2=j\psi+\pi/2$ then using Euler's formula we obtain

$$\sin(j\psi+\pi/2)=\cosh(\psi) \tag{43}$$

and for real ψ , $\cosh(\psi)$ can take on any value from 1 to infinity.

Note that critical angle does not depend on the polarization. Also note that we need to have $n_1>n_2$ to obtain a real value for the critical angle. That is, total internal reflection only occurs when the first medium is “optically denser” than the second medium.

Brewster angle

If $Z_2=Z_1$ then $\rho=0$, there is no reflected field and all incident power is transmitted. For E polarization this would require

$$\eta_1/\cos(\theta_1)=\eta_2/\cos(\theta_2) \tag{44}$$

or

$$\cos^2(\theta_2)=\frac{\epsilon_1\mu_2}{\mu_1\epsilon_2}\cos^2(\theta_1) \tag{45}$$

From Snell's law we know that

$$\sin^2(\theta_2)=\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2(\theta_1) \tag{46}$$

Since $\sin^2(\theta_2)+\cos^2(\theta_2)=1$ we can write

$$\begin{aligned}1&=\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2(\theta_1)+\frac{\epsilon_1\mu_2}{\mu_1\epsilon_2}\cos^2(\theta_1)\\&=\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2(\theta_1)+\frac{\epsilon_1\mu_2}{\mu_1\epsilon_2}[1-\sin^2(\theta_1)]\end{aligned} \tag{47}$$

Solving for $\sin(\theta_1)$ we have

$$\sin(\theta_1)=\sqrt{\frac{\frac{\epsilon_2}{\epsilon_1}-\frac{\mu_2}{\mu_1}}{\frac{\mu_1}{\mu_2}-\frac{\mu_2}{\mu_1}}} \tag{48}$$

For the common case of non-magnetic materials where $\mu_1=\mu_2=\mu_0$ the only solution would be the trivial case $\epsilon_1=\epsilon_2$ where the two media are the same. The problem then reduces to a plane wave propagating in an infinite simple medium.

For H polarization $Z_2=Z_1$ becomes

$$\eta_1\cos(\theta_1)=\eta_2\cos(\theta_2) \tag{49}$$

or

$$\cos^2(\theta_2)=\frac{\epsilon_2\mu_1}{\mu_2\epsilon_1}\cos^2(\theta_1) \tag{50}$$

Using $\sin^2(\theta_2)+\cos^2(\theta_2)=1$ and Snell's law we now have

$$1=\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2(\theta_1)+\frac{\epsilon_2\mu_1}{\mu_2\epsilon_1}[1-\sin^2(\theta_1)] \tag{51}$$

Solving for $\sin(\theta_1)$ we obtain

$$\sin(\theta_1)=\sqrt{\frac{\frac{\mu_2}{\mu_1}-\frac{\epsilon_2}{\epsilon_1}}{\frac{\epsilon_1}{\epsilon_2}-\frac{\epsilon_2}{\epsilon_1}}} \tag{52}$$

If $\mu_2=\mu_1=\mu_0$ the argument of the square root is

$$\frac{1-\frac{\epsilon_2}{\epsilon_1}}{\frac{\epsilon_1}{\epsilon_2}-\frac{\epsilon_2}{\epsilon_1}}=\epsilon_2\frac{\epsilon_1-\epsilon_2}{\epsilon_1^2-\epsilon_2^2}=\frac{\epsilon_2}{\epsilon_1+\epsilon_2} \tag{53}$$

and we have $\theta_1=\theta_B$ where

$$\theta_B=\sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1+\epsilon_2}} \tag{54}$$

is the *Brewster angle*. Note that unlike the critical angle, the Brewster angle is polarization dependent and will exist for both the $\epsilon_1>\epsilon_2$ and $\epsilon_1<\epsilon_2$ cases. At the Brewster angle none of the H polarization is reflected. Consequently, the reflected field can contain only E polarization. Therefore an incident field with arbitrary polarization will be E polarized upon reflection at the Brewster angle. One application of this is the “Brewster window” employed in some laser cavities to maintain a given linear polarization.

Lossy media

Let's now consider the case where medium 1 is lossless (μ_1,ϵ_1 real) while medium 2 is lossy (one or both of μ_2,ϵ_2 complex). We will not use Snell's law at first, but will go back to "first principles." In medium 1 β_x,β_y are real, and medium 2 must have the same β_x,β_y values. Therefore, the z propagation constant must be complex in medium 2. We will write

$$\beta_{c2z}=\beta_{2z}-j\alpha_{2z}=\sqrt{\omega^2\mu_2\epsilon_2-\beta_x^2-\beta_y^2} \tag{55}$$

and the functional dependence of all field vectors in medium 2 will have the form

$$e^{-j\beta_x x}e^{-j\beta_y y}e^{-j\beta_{2z}z}e^{-\alpha_{2z}z} \tag{56}$$

This is a plane wave propagating in a direction determined by the propagation constants $\beta_x,\beta_y,\beta_{2z}$ and exponentially decaying in the z direction. The wave impedance will be complex. For *H* and *E* polarization we have

$$\begin{aligned}Z_H&=\frac{\beta_{2z}-j\alpha_{2z}}{\omega\epsilon_2}\\ Z_E&=\frac{\omega\mu_2}{\beta_{2z}-j\alpha_{2z}}\end{aligned} \tag{57}$$

In lossy media Snell's law still "works" but it not so simple. Since $n_2=\sqrt{\mu_2\epsilon_2/\mu_0\epsilon_0}$ is complex, Snell's law

$$n_1\sin(\theta_1)=n_2\sin(\theta_2) \tag{58}$$

will require a complex angle θ_2. That is, we will need for $\sin(\theta_2)$ to have a complex part to somehow cancel out the complex part of n_2 so the product is equal to the real quantity $n_1\sin(\theta_1)$. What is the sine of a complex angle? Let's write

$$\theta_2=\psi+j\gamma \tag{59}$$

Using a trig identity we have

$$\sin(\psi+j\gamma)=\sin(\psi)\cos(j\gamma)+\cos(\psi)\sin(j\gamma) \tag{60}$$

Using Euler's formula we obtain

$$\begin{aligned}\cos(j\gamma)&=\frac{e^{jj\gamma}+e^{-jj\gamma}}{2}=\cosh(\gamma)\\ \sin(j\gamma)&=\frac{e^{jj\gamma}-e^{-jj\gamma}}{2j}=j\sinh(\gamma)\end{aligned} \tag{61}$$

So

$$\sin(\psi+j\gamma)=\sin(\psi)\cosh(\gamma)+j\cos(\psi)\sinh(\gamma) \tag{62}$$

Snell's law requires

$$\beta_1\sin(\theta_1)=(\beta_2-j\alpha_2)\sin(\psi+j\gamma) \tag{63}$$

or, as a shorthand $\beta_1\sin(\theta_1)=(\beta_2-j\alpha_2)(R+jI)$. The imaginary parts of both sides must be zero, so we have $\beta_2 I=\alpha_2 R$ or $I=(\alpha_2/\beta_2)R$. Then $\beta_1\sin(\theta_1)=\beta_2 R+\alpha_2 I$ which is $\beta_2 R+(\alpha_2^2/\beta_2)R$. Putting these together

$$\beta_1 \sin(\theta_1) = \left(\beta_2 + \frac{\alpha_2^2}{\beta_2}\right) \sin(\psi) \cosh(\gamma)$$
$$\alpha_2 \tan(\psi) = \beta_2 \tanh(\gamma) \tag{64}$$

Clearly it is simpler to use the "first principles" approach in lossy media.

Layered media

For the normal incidence case (previous lecture) we considered general layered media. We obtained reflection and transmission coefficients for the three-media case and an effective impedance formula that could be applied recursively for arbitrary numbers of layers. We could repeat the analysis for the oblique-incidence case, but a little thought shows that we would end up doing exactly the same calculations. The only difference would be the substitutions

$$\eta \to Z$$
$$\beta_k w_k \to (\beta_{kz} - j\alpha_{kz}) w_k \tag{65}$$

where Z is the appropriate wave impedance (57) and

$$\beta_x^2 + \beta_y^2 + (\beta_{kz} - j\alpha_{kz})^2 = \omega^2 \mu_k \epsilon_k \tag{66}$$

In addition, the fields in every layer will have a factor of $e^{-j\beta_x x} e^{-j\beta_y y}$, all with the same β_x, β_y values, as determined by the incident wave. With these substitutions, all of the results derived for the normal-incidence case can be modified to apply to the oblique-incidence case.