Lecture 3c

One-dimensional inhomogeneous media

Introduction

In the previous lectures we have developed tools to analyze layered media in which μ, ϵ have different constant values over various intervals of one coordinate. This is a particular class of inhomogeneous media – piecewise simple media. In that case we can use homogeneous solutions within each simple medium and then apply boundary conditions at each interface. In this lecture we want to investigate some of the subtleties of general inhomogeneous materials. We will limit consideration to source-free, non-magnetic materials for which $\mu=\mu_0$ everywhere.

Inhomogeneous Helmholtz equation

Refer to Equation (42) of Lecture 2a. This is a wave equation for $\mathbf{E}$ assuming constant μ but arbitrary ϵ. If $\mu=\mu_0$, $\epsilon=\epsilon_0\epsilon_r(\mathbf{r})$ and the medium is source-free, this reads

$$\nabla^2\mathbf{E}+\omega^2\mu_0\epsilon_0\epsilon_r(\mathbf{r})\mathbf{E}=\nabla(\nabla\cdot\mathbf{E}) \tag{1}$$

Since

$$\nabla\cdot\mathbf{E}=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z} \tag{2}$$

The right-hand side will be zero if $\mathbf{E}$ has only one component and there is no dependence on the corresponding coordinate. A one-dimensional case would be

$$\mathbf{E}=\hat{a}_x E_x(z) \tag{3}$$

with $\epsilon=\epsilon_0\epsilon_r(z)$. The Helmholtz equation becomes

$$E_x''(z)+\beta_0^2\epsilon_r(z)E_x(z)=0 \tag{4}$$

A two-dimensional case would be

$$\mathbf{E}=\hat{a}_z E_z(x,y) \tag{5}$$

with $\epsilon=\epsilon_0\epsilon_r(x,y)$. The Helmholtz equation is then

$$\nabla^2 E_z(x,y)+\beta_0^2\epsilon_r(x,y)E_z(x,y)=0 \tag{6}$$

In these cases we obtain the "inhomogeneous Helmholtz equation" which is identical to the homogeneous equation with the substation of a spatially varying permittivity.

In general, however, the right-hand side of (1) is not zero. In fact, in a source-free medium $q=0$, so

$$\nabla\cdot(\epsilon\mathbf{E})=\epsilon\nabla\cdot\mathbf{E}+\mathbf{E}\cdot\nabla\epsilon=0 \tag{7}$$

Therefore

$$\nabla\cdot\mathbf{E}=-\mathbf{E}\cdot\frac{\nabla\epsilon}{\epsilon} \tag{8}$$

However, if the permittivity function is very "smooth" then $|\nabla\epsilon/\epsilon|$, which is the maximum fraction change in the permittivity with respect to position, will be very small. If so then $\nabla(\nabla\cdot\mathbf{E})\approx 0$ and

$$\nabla^2\mathbf{E}+\omega^2\mu_0\epsilon_0\epsilon_r(\mathbf{r})\mathbf{E}\approx 0 \tag{9}$$

Therefore, in a "slowly varying" inhomogeneous medium the inhomogeneous Helmholtz equation is approximately valid. In special cases, such as those mentioned above, it is rigorously valid.

One-dimensional inhomogeneous media

If $\epsilon=\epsilon_0\epsilon_r(z)$ and we take $\mathbf{E}=\hat{a}_x E_x(z)$ then we have (rigorously)

$$E_x''(z)+\beta_0^2\epsilon_r(z)E_x(z)=0 \tag{10}$$

This is a 2nd order HLODE and can be solved by the power series methods we discussed at the beginning of the course. Let's consider an example of a linear permittivity profile of the form

$$\epsilon_r(z)=a+bz \tag{11}$$

Then

$$E_x''(z)+[A+Bz]E_x(z)=0 \tag{12}$$

where $A=\beta_0^2 a$ and $B=\beta_0^2 b$. It's convenient to do a change of variable $u=B^{1/3}(z+A/B)$. This gives the simpler equation

$$f''(u)+uf(u)=0 \tag{13}$$

This is a 2nd order HLODE with $p(u)=0$ and $q(u)=u$. These are analytic functions, so we can find solutions of the form

$$f(u)=\sum_{n=0}^{\infty}a_n u^n \tag{14}$$

Substituting into the HLODE we obtain

$$\sum_{n=2}^{\infty}n(n-1)a_n u^{n-2}+\sum_{n=0}^{\infty}a_n u^{n+1}=0 \tag{15}$$

which we can rewrite as

$$\sum_{n=2}^{\infty}n(n-1)a_n u^{n-3}+\sum_{n=3}^{\infty}a_{n-3}u^{n-3}=0 \tag{16}$$

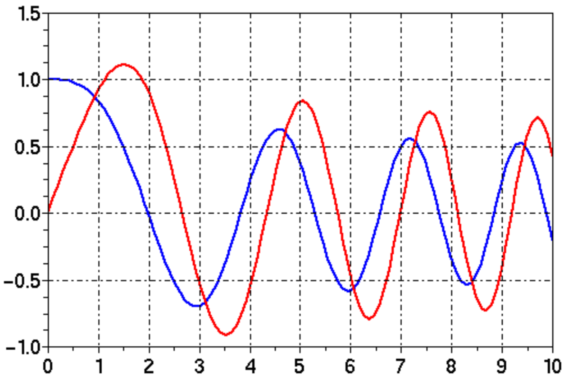
There is one term in the first series with u^{-1} ($n=2$). This gives us the condition $a_2=0$. For all other powers of u we obtain the condition

$$a_n=-\frac{1}{n(n-1)}a_{n-3} \tag{17}$$

With a_0, a_1 arbitrary and $a_2=0$, this specifies all remaining coefficients. This conveniently breaks into two series, one starting with a_0 and one with a_1

$$f_1(u) = \sum_{k=0}^{\infty} a_{3k} u^{3k}$$
$$f_2(u) = \sum_{k=0}^{\infty} a_{3k+1} u^{3k+1} \tag{18}$$

If we take $a_0 = a_1 = 1$, then we obtain the two functions shown in the following plot.



These look somewhat like a cosine and a sine with decreasing amplitudes and decreasing wavelengths.

A general solution will have the form

$$f(u) = c_1 f_1(u) + c_2 f_2(u) \tag{19}$$

Let's use this to solve the problem of reflection and transmission from a smooth transition between two permittivity values. Let's take

$$\epsilon_r = \begin{cases} \epsilon_{r1} & z < -w \\ a + bz & -w \le z \le 0 \\ \epsilon_{r2} & z > 0 \end{cases} \tag{20}$$

where $a = \epsilon_{r2}$ and $b = (\epsilon_{r2} - \epsilon_{r1})/w$. We have $A = \beta_0^2 a$, $B = \beta_0^2 b$ and $u = B^{1/3}(z + A/B)$. If we take the incident electric field to have unit amplitude, then we can write

$$E_x(z) = \begin{cases} e^{-j\beta_1(z+w)} + \rho\, e^{j\beta_1(z+w)} & z < -w \\ c_1 f_1(u) + c_2 f_2(u) & -w \le z \le 0 \\ \tau\, e^{-j\beta_2 z} & z > 0 \end{cases} \tag{21}$$

where ρ, τ are the reflection and transmission coefficients. The boundary conditions are that E_x, H_y are continuous at $z = -w, 0$. From Faraday's law

$$\frac{d}{dz} E_x = -j\omega\mu_0 H_y \tag{22}$$

so the continuity of H_y is equivalent to the continuity of the derivative of E_x. Therefore, defining

$$u_0 = A/B^{2/3}$$
$$u_w = B^{1/3}(-w + A/B) \tag{23}$$

we have at $z = 0$

$$c_1 f_1(u_0) + c_2 f_2(u_0) = \tau$$
$$[c_1 f_1'(u_0) + c_2 f_2'(u_0)]\, B^{1/3} = -j\beta_2 \tau \tag{24}$$

and at $z = -w$

$$1 + \rho = c_1 f_1(u_w) + c_2 f_2(u_w)$$
$$-j\beta_1(1 - \rho) = [c_1 f_1'(u_w) + c_2 f_2'(u_w)]\, B^{1/3} \tag{25}$$

This is four equations in four unknowns. We can put this into matrix form as

$$M = \begin{pmatrix} 0 & -1 & f_1(u_0) & f_2(u_0) \\ 0 & j\beta_2 & f_1'(u_0) B^{1/3} & f_2'(u_0) B^{1/3} \\ -1 & 0 & f_1(u_w) & f_2(u_w) \\ -j\beta_1 & 0 & f_1'(u_w) B^{1/3} & f_2'(u_w) B^{1/3} \end{pmatrix} \tag{26}$$

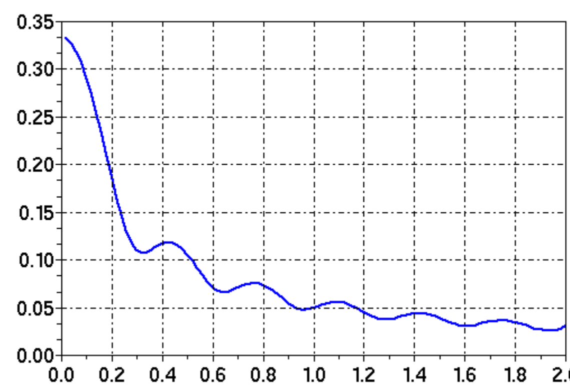
and

$$Y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -j\beta_1 \end{pmatrix} \tag{27}$$

and then solve

$$\begin{pmatrix} \rho \\ \tau \\ c_1 \\ c_2 \end{pmatrix} = M^{-1} Y \tag{28}$$

As an example, if we take $\epsilon_{r1} = 1$, $\epsilon_{r2} = 4$ and $\beta_0 = 2\pi$ (free-space wavelength 1) and vary w from 0 to 2, we get the following result for $|\rho|$



For $w \to 0$ we obtain the value 1/3 corresponding to the piecewise constant case $|\rho| = |(1/2 - 1)/(1/2 + 1)|$. As the transition becomes more gradual, the reflection coefficient drops quite rapidly. This is a general result; smooth transitions produce very small reflected fields. This idea leads to a simple version of the so-called WKB approximation.

WKB approximation

Consider a lossless dielectric with a very smooth (real) permittivity profile $\epsilon=\epsilon_0\epsilon_r(z)$. As we've seen above, very little of the incident energy will be reflected in this case. Let's assume no energy is reflected. We therefore have that the Poynting vector is a constant. Since ϵ varies slowly, we might expect that over small intervals of z the field would behave as a plane wave in a simple medium where $E=H\eta$. We will therefore write

$$\begin{aligned} E_x(z) &= E_0\, a(z)\, e^{-j\phi(z)} \\ H_y(z) &= E_x(z)\frac{1}{\eta(z)} \end{aligned} \tag{29}$$

where $a(z)$ is a real amplitude function and $\phi(z)$ is a real phase function. The (real) impedance is

$$\eta(z)=\eta_0\sqrt{\frac{1}{\epsilon_r(z)}} \tag{30}$$

The Poynting vector points in the z direction and has magnitude

$$P=\frac{1}{2}|E_0|^2\frac{a^2}{\eta} \tag{31}$$

For this to remain constant we must have

$$a(x)=\sqrt{\eta(x)} \tag{32}$$

In a plane wave the phase is $\phi=\beta z$, so $d\phi/dz=\beta_0\sqrt{\epsilon_r}$. Taking

$$\frac{d\phi}{dz}=\beta(z)=\beta_0\sqrt{\epsilon_r(z)} \tag{33}$$

we obtain

$$\phi(z)=\phi_0+\beta_0\int_0^z\sqrt{\epsilon_r(s)}\,ds \tag{34}$$

Equations (29), (32) and (34) are a simple form of the *WKB approximation* for wave propagation in a slowly-varying inhomogeneous medium.

References

1. Chew, W. C., *Waves and Fields in Inhomogeneous Media*, Van Nostrand Reinhold, 1990, ISBN 0-442-23816-9.
2. Bender, C. M. and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, 1978, ISBN 0-07-004452-X.