Lecture 3b

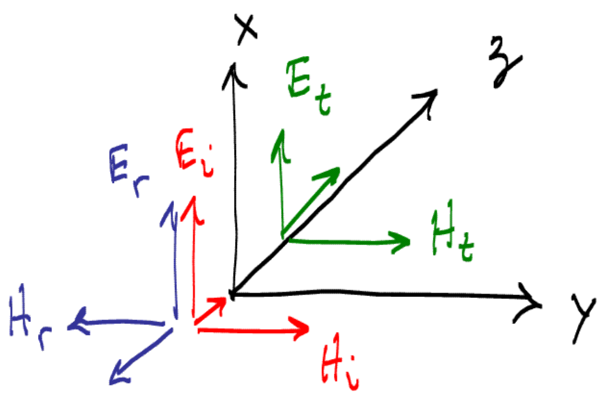
Reflection and transmission at normal incidence

Introduction

We have considered plane wave solutions to Maxwell's equations in a simple medium. Now we consider what happens when a plane wave strikes an interface between two different media. Due to differing impedances on the two sides of the interface, the way cannot propagate from one medium to the next without generating an additional *reflected* wave. This is the simplest example of a *scattering* problem. (In EE 519 we analyze scattering problems in great depth.)

Reflection and transmission coefficients

Let's consider a planar interface between two semi-infinite media. Let the permittivity and permeability be μ_1, ϵ_1 for $z<0$ and μ_2, ϵ_2 for $z\geq 0$. At frequency ω the propagation constants are $\beta_1=\omega\sqrt{\mu_1\epsilon_1}$, $\beta_2=\omega\sqrt{\mu_2\epsilon_2}$ and the intrinsic impedances are $\eta_1=\sqrt{\mu_1/\epsilon_1}$, $\eta_2=\sqrt{\mu_2/\epsilon_2}$. Any or all of these quantities may be complex. Consider the following illustration



For $z\leq 0$ we have an *incident* plane wave

$$\begin{aligned}\mathbf{E}_i &= \hat{a}_x E_0 e^{-j\beta_1 z}\\ \mathbf{H}_i &= \hat{a}_y \frac{E_0}{\eta_1} e^{-j\beta_1 z}\end{aligned} \tag{1}$$

propagating in the $+\hat{a}_z$ direction. For $z\geq 0$ we have a *transmitted* plane wave, also propagating in the $+\hat{a}_z$ direction,

$$\begin{aligned}\mathbf{E}_t &= \hat{a}_x \tau E_0 e^{-j\beta_2 z}\\ \mathbf{H}_t &= \hat{a}_y \tau \frac{E_0}{\eta_2} e^{-j\beta_2 z}\end{aligned} \tag{2}$$

where τ is the *transmission coefficient*. If $\eta_1\neq\eta_2$ there is no way we can have the tangential components of both **E** and **H** match at $z=0$ if we have only these two fields. Instead we require a *reflected* plane wave in $z\leq 0$,

$$\begin{aligned}\mathbf{E}_r &= \hat{a}_x \rho E_0 e^{j\beta_1 z}\\ \mathbf{H}_r &= -\hat{a}_y \rho \frac{E_0}{\eta_1} e^{j\beta_1 z}\end{aligned} \tag{3}$$

that propagates in the $-\hat{a}_z$ direction. Here ρ is the *reflection coefficient*. With these three plane waves, the total electric and magnetic field components are

$$E_x = \begin{cases} E_0 e^{-j\beta_1 z} + \rho E_0 e^{j\beta_1 z} & z\leq 0\\ \tau E_0 e^{-j\beta_2 z} & z\geq 0\end{cases} \tag{4}$$

and

$$H_y = \begin{cases} \dfrac{E_0}{\eta_1} e^{-j\beta_1 z} - \rho \dfrac{E_0}{\eta_1} e^{j\beta_1 z} & z\leq 0\\ \tau \dfrac{E_0}{\eta_2} e^{-j\beta_2 z} & z\geq 0\end{cases} \tag{5}$$

These expressions must be continuous at $z=0$ since there they form the tangential components of **E** and **H**. This gives us the equations

$$\begin{aligned} 1+\rho &= \tau\\ \frac{1-\rho}{\eta_1} &= \frac{\tau}{\eta_2}\end{aligned} \tag{6}$$

The solution is

$$\boxed{\begin{aligned}\rho &= \frac{\eta_2-\eta_1}{\eta_2+\eta_1}\\ \tau &= \frac{2\eta_2}{\eta_2+\eta_1}\end{aligned}} \tag{7}$$

At the interface, the phasors of the reflected and transmitted electric fields are, respectively, ρ and τ times the phasor of the incident electric field. If either of the impedances are complex, then the reflection and/or transmission coefficient may be complex also.

PEC limit

A perfect electric conductor (PEC) would have $\epsilon''\to\infty$. (Remember that the effective conductivity is $\sigma=\omega\epsilon''$.) Let medium 2 be a PEC. Then $\eta_2=\sqrt{\mu_2/\epsilon_2}\to 0$ and it follows that

$$\begin{aligned}\rho &= -1\\ \tau &= 0\end{aligned} \tag{8}$$

There is no transmitted electric field and the reflected field is the negative of the incident field (at the interface). The total electric field on both sides of the interface is zero. We could say that the PEC "shorts out" the electric field.

The total magnetic field at $z=0$ is $H_y=2E_0/\eta_1$. To see that this is true on the $z\geq 0$ side note that $\tau/\eta_2=2/(\eta_2+\eta_1)\to 2/\eta_1$ as $\eta_2\to 0$. However, if medium 2 is

PEC then $\beta_2=\omega\sqrt{\mu_2\epsilon_2}\rightarrow(1-j)\infty$ and the magnetic field goes to zero for any finite value of z.

Interestingly, there is no difference in the ρ,τ values between the cases $\epsilon_2=\epsilon'-j\infty$ (PEC) and $\epsilon_2=\infty-j\epsilon''$ (dielectric with infinite dielectric constant). The important point is that $|\epsilon_2|\rightarrow\infty$ causes $\eta_2\rightarrow 0$.

PMC limit

If $\mu''\rightarrow\infty$ then, by analogy with the PEC case, we call the medium a "*perfect magnetic conductor*" (PMC). Let medium 2 be a PMC. Since $|\eta_2|=\sqrt{\mu_2/\epsilon_2}=\infty$ we have

$$\begin{aligned}\rho&=1\\ \tau&=2\end{aligned} \tag{9}$$

It is easy to verify that $H_y=0$ and $E_x=2E_0$ on both sides of the interface at $z=0$. With $\mu''\rightarrow\infty$ we again get $\beta_2=\omega\sqrt{\mu_2\epsilon_2}\rightarrow(1-j)\infty$ and the electric field goes to zero infinitely fast in medium 2. A PMC "shorts out" the magnetic field.

Another way to get the same values of ρ,τ is to have $\epsilon_1'\rightarrow\infty$ giving $\eta_1=0$ with μ_2,ϵ_2 arbitrary (but finite). This suggests that the interface between a dielectric, with large dielectric constant, and, say, free space, behaves as if free space were a PMC. This is probably the most practical application of the PMC idea, and we will return to it later when we consider dielectric waveguides and resonators in the limit of large dielectric constant.

Power density

At the interface ($z=0$) the Poynting vectors of the three plane waves are

$$\begin{aligned}\mathbf{P}_i&=\ \hat{a}_z\frac{1}{2}\mathrm{Re}\left\{\frac{1}{\eta_1^*}\right\}|E_0|^2\\ \mathbf{P}_r&=-\hat{a}_z\frac{1}{2}\mathrm{Re}\left\{\frac{1}{\eta_1^*}\right\}|E_0|^2|\rho|^2\\ \mathbf{P}_t&=\ \hat{a}_z\frac{1}{2}\mathrm{Re}\left\{\frac{1}{\eta_2^*}\right\}|E_0|^2|\tau|^2\end{aligned} \tag{10}$$

The fraction of incident power that is reflected, P_r/P_i , is given by $|\rho|^2$. The fraction of incident power that is transmitted is *not* given by $|\tau|^2$ due to the different impedance factors in $\mathbf{P}_i,\mathbf{P}_t$. The fraction of incident power that is transmitted is $1-|\rho|^2$, which follows from the conservation of power. Alternately, it is $|\tau|^2[\mathrm{Re}\{1/\eta_2^*\}]/[\mathrm{Re}\{1/\eta_1^*\}]$. If both media are lossless (real impedances) we have $1-|\rho|^2=|\tau|^2\,\eta_1/\eta_2$.

Inverse problem

Given the impedances η_1,η_2 we can calculate the reflection coefficient ρ . We could call this the *forward problem*. Then an interesting (and very practical) variation on this is the *inverse problem*: given η_1,ρ determine η_2 . For example, we may have a material with unknown impedance. We place it in air (known η_1) and we measure the reflected field to determine ρ . We can solve (7) for η_2 to obtain

$$\boxed{\eta_2=\eta_1\frac{1+\rho}{1-\rho}} \tag{11}$$

Therefore a reflection measurement gives us $\eta_2=\sqrt{\mu_2/\epsilon_2}$ and so fixes the ratio of the permeability to the permittivity but does not give us μ_2,ϵ_2 separately. However, if we know that the material is non-magnetic, so $\mu_2=\mu_0$, then we can solve for ϵ_2 as

$$\epsilon_2=\epsilon_1\left(\frac{1-\rho}{1+\rho}\right)^2 \tag{12}$$

This method, or some variation, is typically how one determines permittivity at microwave frequencies. If we need to determine both μ_2,ϵ_2 then an additional measurement is required. A combination of reflection and transmission measurements is an effective way to fully specify the (possibly complex) permittivity and permeability of a sample.

Reflection and transmission in layered media

Let's now consider a slab of material extending over $0\le z\le w$ surrounded by two semi-infinite media so that

$$\mu,\epsilon=\begin{cases}\mu_1,\epsilon_1 & z<0\\ \mu_2,\epsilon_2 & 0\le z\le w\\ \mu_3,\epsilon_3 & z>w\end{cases} \tag{13}$$

For $z<0$ we will have incident and reflected fields.

$$\begin{aligned}\mathbf{E}_1&=\hat{a}_x E_0\left(e^{-j\beta_1 z}+\rho\, e^{j\beta_1 z}\right)\\ \mathbf{H}_1&=\hat{a}_y\frac{E_0}{\eta_1}\left(e^{-j\beta_1 z}-\rho\, e^{j\beta_1 z}\right)\end{aligned} \tag{14}$$

For $z>w$ we will have a transmitted field

$$\begin{aligned}\mathbf{E}_3&=\hat{a}_x E_0\tau\, e^{-j\beta_3(z-w)}\\ \mathbf{H}_3&=\hat{a}_y\frac{E_0}{\eta_3}\tau\, e^{-j\beta_3(z-w)}\end{aligned} \tag{15}$$

Note that by writing the propagation factor as $e^{-j\beta_3(z-w)}$ we are taking the surface $z=w$ as our *phase reference* for the transmitted wave. This the most convenient for algebraic purposes, but we could have used a factor $e^{-j\beta_3 z}$ instead. The result would be an additional factor of $e^{-j\beta_3 w}$ in our expression for τ . In both cases the transmitted field would be the same.

Inside the slab $0 \le z \le w$ we can have plane waves propagating in both the $\pm z$ directions

$$\mathbf{E}_2 = \hat{a}_x E_0 \left(a e^{-j\beta_2 z} + b e^{j\beta_2 z} \right)$$
$$\mathbf{H}_2 = \hat{a}_y \frac{E_0}{\eta_2} \left(a e^{-j\beta_2 z} - b e^{j\beta_2 z} \right) \quad (16)$$

with a and b unknown coefficients. The boundary conditions $\mathbf{E}_1 = \mathbf{E}_2$, $\mathbf{H}_1 = \mathbf{H}_2$ at $z=0$ give us the equations

$$1+\rho = a+b$$
$$\frac{1-\rho}{\eta_1} = \frac{a-b}{\eta_2} \quad (17)$$

We can rewrite these as

$$a+b = (1+\rho)$$
$$a-b = (1-\rho)\frac{\eta_2}{\eta_1} \quad (18)$$

Adding and subtracting allows us to solve for a and b

$$a = \frac{1}{2}\left[(1+\eta_2/\eta_1) + \rho(1-\eta_2/\eta_1) \right]$$
$$b = \frac{1}{2}\left[(1-\eta_2/\eta_1) + \rho(1+\eta_2/\eta_1) \right] \quad (19)$$

The boundary conditions $\mathbf{E}_2 = \mathbf{E}_3$, $\mathbf{H}_2 = \mathbf{H}_3$ at $z=w$ give us the equations

$$a e^{-j\beta_2 w} + b e^{j\beta_2 w} = \tau$$
$$\frac{a e^{-j\beta_2 w} - b e^{j\beta_2 w}}{\eta_2} = \frac{\tau}{\eta_3} \quad (20)$$

Rewriting these we have

$$a e^{-j\beta_2 w} + b e^{j\beta_2 w} = \tau$$
$$a e^{-j\beta_2 w} - b e^{j\beta_2 w} = \frac{\eta_2}{\eta_3}\tau \quad (21)$$

Adding and subtracting these gives us

$$a = \frac{1}{2}\tau e^{j\beta_2 w}(1+\eta_2/\eta_3)$$
$$b = \frac{1}{2}\tau e^{-j\beta_2 w}(1-\eta_2/\eta_3) \quad (22)$$

Equating (19) and (22) eliminates a and b and results in

$$(1+\eta_2/\eta_1) + \rho(1-\eta_2/\eta_1) = \tau e^{j\beta_2 w}(1+\eta_2/\eta_3)$$
$$(1-\eta_2/\eta_1) + \rho(1+\eta_2/\eta_1) = \tau e^{-j\beta_2 w}(1-\eta_2/\eta_3) \quad (23)$$

These are two equations in the two unknowns ρ, τ. Dividing the second equation by the first, we have

$$\frac{(1-\eta_2/\eta_1) + \rho(1+\eta_2/\eta_1)}{(1+\eta_2/\eta_1) + \rho(1-\eta_2/\eta_1)} = e^{-j2\beta_2 w}\frac{1-\eta_2/\eta_3}{1+\eta_2/\eta_3} \quad (24)$$

This is one equation in the single unknown ρ. Multiplying the left side by η_1/η_1 and the right by η_3/η_3 produces

$$\frac{(\eta_1-\eta_2) + \rho(\eta_1+\eta_2)}{(\eta_1+\eta_2) + \rho(\eta_1-\eta_2)} = e^{-j2\beta_2 w}\frac{\eta_3-\eta_2}{\eta_3+\eta_2} \quad (25)$$

On the right we see what looks like a reflection coefficient. Let's define

$$\rho_{12} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1}$$
$$\rho_{23} = \frac{\eta_3-\eta_2}{\eta_3+\eta_2} \quad (26)$$

These are the reflection coefficients for the interfaces $\eta_{1,}\eta_2$ and $\eta_{2,}\eta_3$, respectively. If we divide the numerator and denominator of the left side of (25) by $\eta_1+\eta_1$ we have

$$\frac{-\rho_{12}+\rho}{1-\rho_{12}\rho} = e^{-j2\beta_2 w}\rho_{23} \quad (27)$$

Solving for ρ we obtain

$$\rho = \frac{\rho_{12}+\rho_{23}e^{-j2\beta_2 w}}{1+\rho_{12}\rho_{23}e^{-j2\beta_2 w}} \quad (28)$$

To solve for the transmission coefficient, we can take the first equation of (23) and divide through by $1+\eta_2/\eta_1$ to get

$$\tau e^{j\beta_2 w}\frac{(1+\eta_2/\eta_3)}{(1+\eta_2/\eta_1)} = 1-\rho\rho_{12} \quad (29)$$

so that

$$\tau = e^{-j\beta_2 w}\frac{(1+\eta_2/\eta_1)}{(1+\eta_2/\eta_3)}[1-\rho_{12}\rho] \quad (30)$$

Since

$$1-\rho_{12}\rho = \frac{1+\rho_{12}\rho_{23}e^{-j2\beta_2 w}}{1+\rho_{12}\rho_{23}e^{-j2\beta_2 w}} - \rho_{12}\frac{\rho_{12}+\rho_{23}e^{-j2\beta_2 w}}{1+\rho_{12}\rho_{23}e^{-j2\beta_2 w}}$$
$$= \frac{1-\rho_{12}^2}{1+\rho_{12}\rho_{23}e^{-j2\beta_2 w}} \quad (31)$$

we obtain

$$\tau = e^{-j\beta_2 w}\frac{(1+\eta_2/\eta_1)}{(1+\eta_2/\eta_3)}\frac{1-\rho_{12}^2}{1+\rho_{12}\rho_{23}e^{-j2\beta_2 w}} \quad (32)$$

An important special case is when $\eta_3 = \eta_1$ and medium 2 is a "window" of width w embedded in medium 1. In this case $\rho_{23} = -\rho_{21}$ and

$$\rho = \rho_{12}\frac{1-e^{-j2\beta_2 w}}{1-\rho_{12}^2 e^{-j2\beta_2 w}}$$
$$\tau = e^{-j\beta_2 w}\frac{1-\rho_{12}^2}{1-\rho_{12}^2 e^{-j2\beta_2 w}} \quad (33)$$

If we measure ρ,τ then these become two equations in the two unknowns ρ_{12} and $e^{-j\beta_2 w}$. Solving for these provides us with the values of $\sqrt{\mu_2/\epsilon_2}$ (from η_2) and $\sqrt{\mu_2\epsilon_2}$ (from β_2). Know their ratio and product we can solve for $\mu_{2,}\epsilon_2$ separately.

Weak-scattering limit

An important special case is the *weak scattering limit* in which the differences of the three impedances is small. This implies that both ρ_{12},ρ_{23} are small. Therefore the $\rho_{12}\rho_{23}$ factor in the denominator (28) is negligible and the reflection coefficient simplifies to

$$\rho \approx \rho_{12}+\rho_{23}e^{-j2\beta_2 w} \tag{34}$$

Some of the incident field is reflected at $z=0$ (the ρ_{12} term). The field propagates a distance *w* in medium 2, strikes the second interface and generates a second reflected field (the ρ_{23} term). This propagates *w* back through medium 2 and adds to the first reflection. The round trip through medium 2 accounts for the $e^{-j2\beta_2 w}$ factor.

Of course the second reflected field would also generate an additional reflection traveling in the +*z* direction and there would in fact be an infinite series of reflections at the two interfaces getting progressively smaller. In the weak scattering approximation we neglect this *multiple scattering* effect. In the exact expression (28) it is accounted for by the denominator.

Effective impedance

Suppose you measured the reflection coefficient given in (28) and you assumed it was due to a single interface at $z=0$ between materials with impedances η_1 and η. From $\rho=(\eta-\eta_1)/(\eta+\eta_1)$ you could solve for η as

$$\eta=\eta_1\frac{1+\rho}{1-\rho} \tag{35}$$

Now, from (28)

$$1\pm\rho=\frac{1+\rho_{12}\rho_{23}e^{-j2\beta_2 w}\pm[\rho_{12}+\rho_{23}e^{-j2\beta_2 w}]}{1+\rho_{12}\rho_{23}e^{-j2\beta_2 w}} \tag{36}$$

Substituting this into our expression for η we get

$$\begin{aligned}\eta&=\eta_1\frac{1+\rho_{12}\rho_{23}e^{-j2\beta_2 w}+[\rho_{12}+\rho_{23}e^{-j2\beta_2 w}]}{1+\rho_{12}\rho_{23}e^{-j2\beta_2 w}-[\rho_{12}+\rho_{23}e^{-j2\beta_2 w}]}\\&=\eta_1\frac{(1+\rho_{12})(1+\rho_{23}e^{-j2\beta_2 w})}{(1-\rho_{12})(1-\rho_{23}e^{-j2\beta_2 w})}\\&=\eta_1\frac{\eta_2}{\eta_1}\frac{1+\rho_{23}e^{-j2\beta_2 w}}{1-\rho_{23}e^{-j2\beta_2 w}}\end{aligned} \tag{37}$$

where in the last line we used $\eta_2=\eta_1(1+\rho_{12})/(1-\rho_{12})$. If we substitute

$$\rho_{23}=\frac{\eta_3-\eta_2}{\eta_3+\eta_2} \tag{38}$$

and multiply numerator and denominator by $\eta_3+\eta_2$, we obtain

$$\begin{aligned}\eta&=\eta_2\frac{(\eta_3+\eta_2)+(\eta_3-\eta_2)e^{-j2\beta_2 w}}{(\eta_3+\eta_2)-(\eta_3-\eta_2)e^{-j2\beta_2 w}}\\&=\eta_2\frac{\eta_3(1+e^{-j2\beta_2 w})+\eta_2(1-e^{-j2\beta_2 w})}{\eta_2(1+e^{-j2\beta_2 w})+\eta_3(1-e^{-j2\beta_2 w})}\end{aligned} \tag{39}$$

Finally, using

$$\begin{aligned}1+e^{-j2\beta_2 w}&=e^{-j\beta_2 w}2\cos(\beta_2 w)\\1-e^{-j2\beta_2 w}&=e^{-j\beta_2 w}2j\sin(\beta_2 w)\end{aligned} \tag{40}$$

we arrive at

$$\eta=\eta_2\frac{\eta_3\cos(\beta_2 w)+j\eta_2\sin(\beta_2 w)}{\eta_2\cos(\beta_2 w)+j\eta_3\sin(\beta_2 w)} \tag{41}$$

This is the *effective impedance* of the combination of media 2 and 3. That is, a single medium with this impedance will give the same reflection coefficient as the combination of media 2 and 3.

The effective impedance expression simplifies in the case $w\to 0$. Using 1st-order Taylor series for cosine and sine we have

$$\begin{aligned}\eta&\approx\eta_2\frac{\eta_3+j\eta_2\beta_2 w}{\eta_2+j\eta_3\beta_2 w}\\&\approx\eta_3+j\beta_2 w\left(\eta_2-\frac{\eta_3^2}{\eta_2}\right)\end{aligned} \tag{42}$$

In the last line we used the Taylor series $1/(1+x)\approx 1-x$.

Impedance matching

An interesting special case is when the equivalent impedance is $\eta=\eta_1$ since this gives $\rho=0$. We then say that the slab has performed *impedance matching*. Let's consider the important case where media 1,2,3 are all lossless so η_1,η_2,η_3 are real. If $\cos(\beta_2 w)=0$ then

$$\eta=\frac{\eta_2^2}{\eta_3}=\eta_1 \tag{43}$$

if

$$\eta_2=\sqrt{\eta_1\eta_3} \tag{44}$$

The smallest *w* that gives $\cos(\beta_2 w)=0$ corresponds to $\beta_2 w=\pi/2$ or $w=\lambda_2/4$. That is, the slab is one-quarter of a wavelength (in medium 2) thick and its impedance is the geometric mean of the impedances of media 1 and 3. This is a *quarter-wave matching layer*.

Another matching possibility is $\sin(\beta_2 w)=0$. Then

$$\eta=\eta_3=\eta_1 \tag{45}$$

if $\eta_3=\eta_1$, that is, if media 1 and 3 are the same. The smallest *w* (other than the trivial case $w=0$) that gives $\sin(\beta_2 w)=0$ is $\beta_2 w=\pi$ or $w=\lambda_2/2$. Thus a dielectric window of half-wavelength thickness (in the medium) is transparent.

Multi-layered media

Consider a material consisting of $N-1$ slabs with impedances $\eta_{1,}\ldots,\eta_{N-1}$, propagation constants $\beta_{1,}\ldots,\beta_{N-1}$ and widths $w_{1,}\ldots,w_{N-1}$. Assume these are layered on top of a semi-infinite material with impedance η_N . To get the effective impedance of the entire structure we can start with $\eta_N^e=\eta_N$ and then iterate using

$$\eta_k^e=\eta_k^e\frac{\eta_{k+1}^e\cos(\beta_k w_k)+j\eta_k^e\sin(\beta_k w_k)}{\eta_k^e\cos(\beta_k w_k)+j\eta_{k+1}^e\sin(\beta_k w_k)} \tag{46}$$

until we reach the effective impedance at the first interface, η_1^e . The reflection coefficient is then

$$\rho=\frac{\eta_1^e-\eta_0}{\eta_1^e+\eta_0} \tag{47}$$

assuming the incident wave is traveling in medium with impedance η_0 .

References

1. http://www.mellesgriot.com/products/optics/oc_1.htm