Lecture 3a

Plane waves in simple media

Helmholtz equation in rectangular coordinates

In rectangular coordinates the (scalar) Helmholtz equation

$$\nabla^2 A+\beta^2 A=0 \tag{1}$$

reads

$$\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}+\frac{\partial^2 A}{\partial z^2}+\beta^2 A=0 \tag{2}$$

Here A could represent any component of **A**, **F**, **E**, **H** in a source-free simple medium and $\beta^2=\omega^2\mu\epsilon$. Let's consider the solution of this equation.

Separation of variables

The Helmholtz equation is an example of a *partial differential equation* (PDE), an equation relating the partial derivatives of a function of several variables. In general PDE's are more difficult to solve than ODE's, and there are fewer systematic techniques available. One of these is called *separation of variables*. This technique only works in certain coordinate systems, but when it does it converts a PDE into a system of ODE's which we can then solved one-by-one using standard techniques.

The essence of separation of variables is to represent a three-dimensional function of the variables *u,v,w* as a product of three one-dimensional functions: $A(u,v,w)=f(u)g(v)h(w)$. We substitute this into the PDE and then attempt to collect all terms containing the u variable on one side of the equation. This results in an equation of the form $B(u)=C(v,w)$. Since $C(v,w)$ does not depend on u, $B(u)$ does not either. This means that $B(u)=k_u$ where k_u is a constant. This provides us with an ODE in the variable u, and the solution is the function $f(u)$. We repeat the process for the v and w variables to obtain the functions $g(u)$ and $h(u)$.

In rectangular coordinates we look for a solution of the form $A(x,y,z)=f(x)g(y)h(z)$. Substituting into the Helmholtz equation results in

$$f''gh+fg''h+fgh''+\beta^2 fgh=0 \tag{3}$$

where $f''=d^2f/dx^2$, $g''=d^2g/dy^2$, and $h''=d^2h/dz^2$. Diving by fgh gives us

$$\frac{f''}{f}+\frac{g''}{g}+\frac{h''}{h}+\beta^2=0 \tag{4}$$

and we can then write

$$\frac{f''}{f}=-\left(\beta^2+\frac{g''}{g}+\frac{h''}{h}\right) \tag{5}$$

The left-hand side is a function of x only. The right-hand side is a function of y and z only. Therefore we have

$$\frac{f''}{f}=-\beta_x^2 \tag{6}$$

By calling our constant $-\beta_x^2$ we are using a little foresight. As we will see, in many cases of interest f''/f is a negative real number, and β_x will be a positive number representing a *propagation constant*. However, we could take β_x to be any complex number, so $-\beta_x^2$ is completely arbitrary.

We could have isolated g''/g or h''/h instead giving us

$$\begin{aligned}\frac{g''}{g}&=-\beta_y^2\\ \frac{h''}{h}&=-\beta_z^2\end{aligned} \tag{7}$$

For equation (4) to be satisfied we require

$$\beta_x^2+\beta_y^2+\beta_z^2=\beta^2 \tag{8}$$

Now, the general solution of

$$f''+\beta_x^2 f=0 \tag{9}$$

can be expressed in the form

$$f=ae^{-j\beta_x x}+be^{j\beta_x x} \tag{10}$$

or in the form

$$f=a\cos(\beta_x x)+b\sin(\beta_x x) \tag{11}$$

and likewise the y and z functions. We will use the following compact notation to represent the separation of variables solution

$$A=\begin{Bmatrix}e^{-j\beta_x x}\\ e^{j\beta_x x}\end{Bmatrix}\begin{Bmatrix}e^{-j\beta_y y}\\ e^{j\beta_y y}\end{Bmatrix}\begin{Bmatrix}e^{-j\beta_z z}\\ e^{j\beta_z z}\end{Bmatrix} \tag{12}$$

The braces denote an arbitrary linear combination of the functions inside. Remember that the constants β_x,β_y,β_z can be any complex numbers. The only constraint is $\beta_x^2+\beta_y^2+\beta_z^2=\beta^2$.

Using separation of variables we have found a certain family of solutions. But, that doesn't mean that all solutions of the Helmholtz equation are of the form $f(x)g(y)h(z)$. What about solutions that are not separable? Just as a one-dimensional function can be represented as an inverse Fourier transform, a three-dimensional can be represented as an inverse three-dimensional Fourier transform of the form

$$A(x,y,z)=\frac{1}{(2\pi)^3}\iiint S(\beta_x,\beta_y,\beta_z)\left[e^{-j(\beta_x x+\beta_y y+\beta_z z)}\right]d\beta_x d\beta_y d\beta_z \tag{13}$$

The product in the brackets is one of our separation of variables type solutions. So, although there are solutions of the Helmholtz equation which are not of the form (12), all solutions can be represented as a superposition of functions of the form (12).

Plane waves and intrinsic impedance

Let's consider a field in a source-free simple medium that depends on *z* only. The Helmholtz equation reduces to

$$\frac{\partial^2 A}{\partial z^2}+\beta^2 A=0 \tag{14}$$

A general solution is a linear combination of $e^{-j\beta z}$ and $e^{j\beta z}$. Let's consider only $e^{-j\beta z}$ dependence for now. In a source-free medium Maxwell's equations are

$$\mathbf{H}=\frac{j}{\omega\mu}\nabla\times\mathbf{E} \qquad \mathbf{E}=\frac{-j}{\omega\epsilon}\nabla\times\mathbf{H} \tag{15}$$

The *z* component of **H** is

$$H_z=\frac{j}{\omega\mu}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right) \tag{16}$$

But, we are assuming there is no *x* or *y* dependence, so $H_z=0$. Similarly $E_z=0$. This shows that the field is *transverse* to the *z* axis, and our solution will have the form

$$\begin{aligned}E_x&=E_{x0}e^{-j\beta z}\\E_y&=E_{y0}e^{-j\beta z}\\H_x&=H_{x0}e^{-j\beta z}\\H_y&=H_{y0}e^{-j\beta z}\end{aligned} \tag{17}$$

where the constants E_{x0} etc. are arbitrary complex numbers. From (15)

$$\begin{aligned}H_y&=\frac{j}{\omega\mu}\frac{\partial E_x}{\partial z}\\&=\frac{\beta}{\omega\mu}E_{x0}e^{-j\beta z}\end{aligned} \tag{18}$$

so

$$H_{y0}=\frac{\beta}{\omega\mu}E_{x0} \tag{19}$$

or

$$H_{y0}=\frac{1}{\eta}E_{x0} \tag{20}$$

where

$$\eta=\frac{\omega\mu}{\beta}=\frac{\omega\mu}{\omega\sqrt{\mu\epsilon}}=\sqrt{\frac{\mu}{\epsilon}} \tag{21}$$

is the *intrinsic impedance* of the medium. It has units of ohms. Note that in a lossless medium, with μ,ϵ real, η is real. In a lossy medium it will in general be complex. Similarly we find

$$H_{x0}=-\frac{1}{\eta}E_{y0} \tag{22}$$

Calling

$$\begin{aligned}\mathbf{E}_0&=\hat{a}_x E_{x0}+\hat{a}_y E_{y0}\\\mathbf{H}_0&=\hat{a}_x H_{x0}+\hat{a}_y H_{y0}\end{aligned} \tag{23}$$

we see that

$$\mathbf{H}_0=\frac{1}{\eta}\hat{a}_z\times\mathbf{E}_0 \tag{24}$$

Our solution is

$$\begin{aligned}\mathbf{E}&=\mathbf{E}_0 e^{-j\beta z}\\\mathbf{H}&=\mathbf{H}_0 e^{-j\beta z}\end{aligned} \tag{25}$$

This is called a *plane wave* since the fields are the same at all points of the plane $z=\text{const}$. The intrinsic impedance of the medium

$$\boxed{\eta=\sqrt{\frac{\mu}{\epsilon}}} \tag{26}$$

depends only on the permeability and permittivity and will be complex if those parameters are. The impedance of free space is approximately $377\,\Omega$.

Consider the general case where $\epsilon_c=\epsilon'-j\epsilon''$ and $\mu_c=\mu'-j\mu''$. The propagation constant will be complex, and we write $\beta_c=\omega\sqrt{\mu_c\epsilon_c}=\beta-j\alpha$. We then have, for example,

$$\begin{aligned}E_x(z,t)&=\text{Re}\{E_{x0}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\}\\&=|E_{x0}|e^{-\alpha z}\cos(\omega t-\beta z+\angle E_{x0})\end{aligned} \tag{27}$$

For a time change Δt the argument of the cosine remains constant if the position change is $\Delta z=(\omega/\beta)\Delta t$. Our field therefore propagates along the *z* axis with *phase velocity*

$$v_p=\frac{\omega}{\beta}=\frac{1}{\text{Re}\sqrt{\mu_c\epsilon_c}} \tag{28}$$

The phase velocity in free space is the *speed of light*

$$c=\frac{1}{\sqrt{\mu_c\epsilon_0}}\equiv 299{,}792{,}458\text{ m/s} \tag{29}$$

The distance between peaks of the cosine is the *wavelength*

$$\lambda=\frac{2\pi}{\beta}=\frac{2\pi}{\text{Re}\{\beta_c\}} \tag{30}$$

A plane wave propagating in an arbitrary direction can be specified as follows. Choose a direction of propagation $\hat{a}_P$ and choose any electric field phasor $\mathbf{E}_0$ that is orthogonal to $\hat{a}_P$, that is,

$$\hat{a}_P\cdot\mathbf{E}_0=0 \tag{31}$$

The magnetic field phasor is

$$\mathbf{H}_0=\frac{1}{\eta_c}\hat{a}_P\times\mathbf{E}_0 \tag{32}$$

and the fields are

$$\mathbf{E}=\mathbf{E}_0 e^{-\alpha\hat{a}_P\cdot\mathbf{r}} e^{-j\beta\hat{a}_P\cdot\mathbf{r}}$$
$$\mathbf{H}=\mathbf{H}_0 e^{-\alpha\hat{a}_P\cdot\mathbf{r}} e^{-j\beta\hat{a}_P\cdot\mathbf{r}} \tag{33}$$

where $\beta-j\alpha=\omega\sqrt{\mu_c\epsilon_c}$. The time-average Poynting vector is

$$\mathbf{P}=\frac{1}{2}\mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^*\right)$$
$$=\hat{a}_P e^{-2\alpha\hat{a}_P\cdot\mathbf{r}}\frac{1}{2}\mathrm{Re}\left\{\frac{1}{\eta_c^*}\right\}E_0^2 \tag{34}$$

If the medium is lossy ($\alpha\neq 0$) then the power density of a plane wave will decay by $e^{-2\alpha L}$ after propagating a distance *L*. Defining the *skin depth* of a material to be

$$\delta=\frac{1}{\alpha} \tag{35}$$

the power density decreases by a factor of $e^{-2}\approx 0.135$ after propagating a distance δ.

Propagation constant

For a general simple medium we have

$$\beta_c=\beta-j\alpha=\omega\sqrt{\mu_c\epsilon_c} \tag{36}$$

This is in general a complex number. It's magnitude squared is

$$|\beta_c|^2=\beta^2+\alpha^2=\omega^2|\mu_c\epsilon_c| \tag{37}$$

We also have

$$\mathrm{Re}[\beta_c^2]=\beta^2-\alpha^2=\omega^2\mathrm{Re}[\mu_c\epsilon_c] \tag{38}$$

By adding and subtracting these two equations we can solve for

$$\beta=\frac{\omega}{\sqrt{2}}\sqrt{|\mu_c\epsilon_c|+\mathrm{Re}[\mu_c\epsilon_c]}$$
$$\alpha=\frac{\omega}{\sqrt{2}}\sqrt{|\mu_c\epsilon_c|-\mathrm{Re}[\mu_c\epsilon_c]} \tag{39}$$

The second expression makes clear that if μ,ϵ are both real then $\alpha=0$ and the medium is lossless.

Most media that do not contain ferromagnetic materials (iron, cobalt, etc.) have a permeability essentially equal to that of free space. For such a *non-magnetic* medium we have

$$\mu_c=\mu_0$$
$$\epsilon_c=\epsilon'-j\epsilon'' \tag{40}$$

so

$$\sqrt{|\mu_c\epsilon_c|\pm\mathrm{Re}[\mu_c\epsilon_c]}=\sqrt{\mu_0\epsilon'}\sqrt{\sqrt{1+(\epsilon''/\epsilon')^2}\pm 1} \tag{41}$$

The double-square-root expression, with $x=\epsilon''/\epsilon'$, has the limiting behavior

$$\sqrt{\sqrt{1+x^2}+1}\approx\begin{cases}\sqrt{2} & \text{for } x\ll 1\\ \sqrt{x} & \text{for } x\gg 1\end{cases}$$
$$\sqrt{\sqrt{1+x^2}-1}\approx\begin{cases}\dfrac{x}{\sqrt{2}} & \text{for } x\ll 1\\ \sqrt{x} & \text{for } x\gg 1\end{cases} \tag{42}$$

These two limits correspond to low-loss and high-loss media. The dimensionless ratio ϵ''/ϵ' is called the *loss tangent*. Instead of specifying ϵ' and ϵ'' it is often more useful to specify the *dielectric constant* $\epsilon_r'=\epsilon'/\epsilon_0$ and the *loss tangent* $\tan\Delta=\epsilon''/\epsilon'$.

Low-loss medium ("good dielectric")

This is the case of small loss tangent, $\epsilon''/\epsilon'\ll 1$. Using the $x\ll 1$ expressions from above we can write

$$\beta=\frac{\omega}{\sqrt{2}}\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon_r'}\sqrt{2} \tag{43}$$

and

$$\alpha=\frac{\omega}{\sqrt{2}}\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon_r'}\frac{\epsilon''/\epsilon'}{\sqrt{2}} \tag{44}$$

so

$$\beta=\beta_0\sqrt{\epsilon_r'}$$
$$\alpha=\beta_0\sqrt{\epsilon_r'}\frac{(\epsilon''/\epsilon')}{2} \tag{45}$$

where $\beta_0=\omega\sqrt{\mu_0\epsilon_0}=\omega/c$ is the propagation constant in free space at the given frequency. In a low-loss medium $\alpha\ll\beta$ so a wave will travel many wavelengths ($\lambda=2\pi/\beta$) before it is significantly attenuated. Another way to express this is $\lambda\gg\delta$. Note that $\alpha/\beta=(\epsilon''/\epsilon')/2$ so the loss tangent is $2\alpha/\beta$.

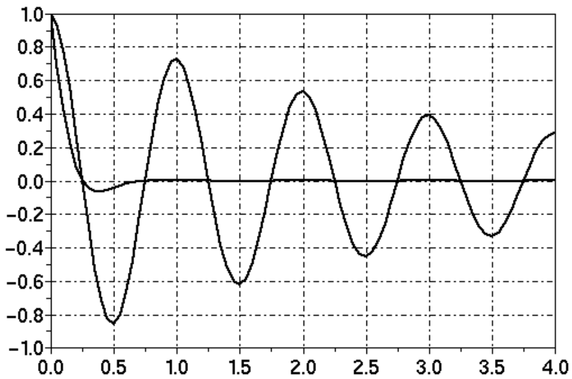
High-loss medium ("good conductor")

This is the case of high loss tangent, $\epsilon''/\epsilon'\gg 1$. Using the $x\gg 1$ expressions we find

$$\alpha=\beta=\beta_0\sqrt{\epsilon_r'}\sqrt{\frac{\epsilon''/\epsilon'}{2}} \tag{46}$$

In a high-loss medium $\alpha=\beta$ so a wave will experience significant loss over a single wavelength. We have $\lambda=2\pi\delta$.

The following plot shows the electric field amplitude for two cases. In both $\beta=2\pi$, $\lambda=1$. The slowly decaying curve has $\alpha=\beta/10$ while the other curve has $\alpha=\beta$.



Polarization

Going back to a plane wave propagating in the z direction, the electric field is

$$\mathbf{E}=\mathbf{E}_0 e^{-j\beta z} \quad (47)$$

with

$$\mathbf{E}_0=\hat{a}_x E_{x0}+\hat{a}_y E_{y0} \quad (48)$$

Let's see how this field behaves through time at $z=0$. We have

$$\begin{aligned} E_x(t) &= \mathrm{Re}\left\{E_{x0} e^{j\omega t}\right\}=|E_{x0}|\cos(\omega t+\angle E_{x0}) \\ E_y(t) &= \mathrm{Re}\left\{E_{y0} e^{j\omega t}\right\}=|E_{y0}|\cos(\omega t+\angle E_{y0}) \end{aligned} \quad (49)$$

Subtracting $\angle E_{x0}$ from each argument (equivalent to a change of time reference) we have

$$\begin{aligned} E_x(t) &= |E_{x0}|\cos(\omega t) \\ E_y(t) &= |E_{y0}|\cos(\omega t+\phi) \end{aligned} \quad (50)$$

where $\phi=\angle E_{y0}-\angle E_{x0}$. There are two important special cases.

Linear polarization

If $\phi=0,\pi$ then

$$\begin{aligned} E_x(t) &= \ |E_{x0}|\cos(\omega t) \\ E_y(t) &= \pm|E_{y0}|\cos(\omega t) \end{aligned} \quad (51)$$

and $E_y(t)/E_x(t)=\pm|E_{y0}|/|E_{x0}|$. Over time the electric field traces out a line with slope $\pm|E_{y0}|/|E_{x0}|$. This is the case of *linear polarization*. We can use any two orthogonal directions as the x and y axes. For terrestrial applications we usually use the ground as a reference and talk about *vertical* (V) and *horizontal* (H) polarization.

Circular polarization

If $\phi=\mp\pi/2$ and $|E_{y0}|=|E_{x0}|=|E_0|$ then

$$\begin{aligned} E_x(t) &= \ |E_0|\cos(\omega t) \\ E_y(t) &= \pm|E_0|\sin(\omega t) \end{aligned} \quad (52)$$

This is the case of *circular polarization* where the electric field traces out a circle. The rotation can be in either of two directions. If $E_y(t)=|E_0|\sin(\omega t)$ we have *right-hand* polarization. If you point your right thumb in the direction of propagation, the electric field rotates in the direction of your curled fingers. Conversely if $E_y(t)=-|E_0|\sin(\omega t)$ we have *left-hand* polarization.

Elliptical polarization

Any case other than linear or circular polarization is called *elliptical polarization* because the electric field traces out an ellipse. The ellipse has an *axial ratio*

$$\frac{|\mathbf{E}_{max}|}{|\mathbf{E}_{min}|}=\pm\cot(\epsilon) \quad (53)$$

where

$$\sin(2\epsilon)=\frac{2|E_{x0}||E_{y0}|}{|E_{x0}|^2+|E_{y0}|^2}\sin(\phi) \quad (54)$$

The *tilt angle* τ between the major axis of the ellipse and the x axis is given by

$$\tan(2\tau)=\tan(2\alpha)\cos(\phi) \quad (55)$$

where

$$\tan(\alpha)=\frac{|E_{y0}|}{|E_{x0}|} \quad (56)$$

(see the Mott reference for details).

References

1. Mott, H., *Antennas for Radar and Communications: A Polarimetric Approach*, Wiley, 1992, ISBN 0-471-57538-0. (Section 3.3)