Lecture 2d

Electromagnetic theorems

Introduction

Most of this course will be devoted to solving Maxwell's equations (usually as expressed in the Helmholtz equation) in different coordinate systems and with different boundary conditions. However, there are some important general results we can derive from Maxwell's equations without reference to a particular coordinate system or boundary conditions. We will derive a few of these important electromagnetic theorems in this lecture.

Poynting's theorem

In many applications it is of great importance to be able to quantitatively describe how an EM field carries energy from one point to another. This is the basis of vision, radio, fiber optics and many, many fundamental phenomena and technologies.

The units of **E** and **H** are V/m and A/m, respectively. A product of **E** and **H** will therefore have units of $\mathrm{W/m^2}$ which corresponds to a power density, or intensity. If the product produces a vector it might correspond to power flow since there will be a direction associated with the $\mathrm{W/m^2}$. This is a clue that $\mathbf{E}\times\mathbf{H}$, or something closely related, might be what we are looking for.

Consider

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}^*)=\mathbf{H}^*\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\nabla\times\mathbf{H}^* \tag{1}$$

The divergence term will allow us to use the divergence theorem. The conjugate will allow us to end up with terms like $\mathbf{E}\cdot\mathbf{E}^*=E^2$ which are related to energy density. Faraday's law and the conjugate of Ampere's law are

$$\begin{aligned}\nabla\times\mathbf{E}&=-j\omega\mu_c\mathbf{H}\\ \nabla\times\mathbf{H}^*&=\mathbf{J}_i^*-j\omega\epsilon_c^*\mathbf{E}^*\end{aligned} \tag{2}$$

where we explicitly allow for complex permittivity and permeability. Any conductivity present is accounted for in the complex permittivity. Substitution of this into the previous equations gives

$$\begin{aligned}\nabla\cdot(\mathbf{E}\times\mathbf{H}^*)&=\mathbf{H}^*\cdot(-j\omega\mu_c\mathbf{H})-\mathbf{E}\cdot(\mathbf{J}_i^*-j\omega\epsilon_c^*\mathbf{E}^*)\\ &=-j\omega\mu_c H^2+j\omega\epsilon_c^*E^2-\mathbf{E}\cdot\mathbf{J}_i^*\end{aligned} \tag{3}$$

From the divergence theorem we have

$$\iiint\nabla\cdot(\mathbf{E}\times\mathbf{H}^*)dv=\oint\!\!\oint(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{ds} \tag{4}$$

Therefore

$$\oint\!\!\oint(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{ds}=\iiint[-j\omega\mu_c H^2+j\omega\epsilon_c^*E^2-\mathbf{E}\cdot\mathbf{J}_i^*]dV \tag{5}$$

or

$$\begin{aligned}\oint\!\!\oint(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{ds}=&-\iiint(\mathbf{E}\cdot\mathbf{J}_i^*)dv\\ &-\iiint j\omega(\mu'-j\mu'')H^2dv\\ &+\iiint j\omega(\epsilon'+j\epsilon'')E^2dv\end{aligned} \tag{6}$$

Taking 1/2 the real part of each side, we obtain

$$\begin{aligned}&\oint\!\!\oint\frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{ds}+\iiint\frac{1}{2}\mathrm{Re}(\mathbf{E}\cdot\mathbf{J}_i^*)dv\\ &\quad+\frac{1}{2}\iiint\omega\mu''H^2dv+\frac{1}{2}\iiint\omega\epsilon''E^2dv=0\end{aligned} \tag{7}$$

Identifying $\sigma=\omega\epsilon''$, $\sigma_m=\omega_m\mu''$ and rearranging we have

$$\begin{aligned}-\iiint_V\frac{1}{2}\mathrm{Re}(\mathbf{E}\cdot\mathbf{J}_i^*)dV=&\oint\!\!\oint_S\frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{ds}\\ &+\frac{1}{2}\iiint_V\sigma_m H^2dV+\frac{1}{2}\iiint_V\sigma E^2dV\end{aligned} \tag{8}$$

We recognize three of these terms:

$$\begin{aligned}\text{power supplied by }\mathbf{J}_i=&\oint\!\!\oint_S\frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{ds}\\ &+\text{power lost in magnetic conduction}\\ &+\text{power lost in electric conduction}\end{aligned} \tag{9}$$

The powers in these cases are all "time averaged." Other than conduction losses, the only place for the power supplied by the impressed current to go is to be carried through the surface *S* by the fields. Therefore we conclude that

$$\oint\!\!\oint_S\frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{ds}=\text{power flow through }S \tag{10}$$

This must be true for any surface *S*. We are led to define the *Poynting vector* **P** as

$$\boxed{\mathbf{P}=\frac{1}{2}\mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^*\right)} \tag{11}$$

This represents the power per unit area carried by the field.

Uniqueness theorem

There are infinite possible solutions to Maxwell's equations - any physically possible EM fields. In order to have a problem with a unique solution we must specify additional constraints on the fields. The *uniqueness theorem* tells us how to do this.

Consider two sets of fields $(\mathbf{E}_1,\mathbf{H}_1)$ and $(\mathbf{E}_2,\mathbf{H}_2)$ that satisfy Maxwell's equations with the same (possibly complex) distribution of permittivity and permeability and the same impressed current throughout some volume *V*:

$$\begin{aligned}\nabla\times\mathbf{E}_1&=-j\omega\mu_c\mathbf{H}_1\\ \nabla\times\mathbf{H}_1&=\mathbf{J}_i+j\omega\epsilon_c\mathbf{E}_1\end{aligned} \tag{12}$$

and

$$\begin{aligned}\nabla\times\mathbf{E}_2&=-j\omega\mu_c\mathbf{H}_2\\ \nabla\times\mathbf{H}_2&=\mathbf{J}_i+j\omega\epsilon_c\mathbf{E}_2\end{aligned} \tag{13}$$

Subtract the second set of equations from the first set and note that $\nabla\times\mathbf{E}_1-\nabla\times\mathbf{E}_2=\nabla\times\delta\,\mathbf{E}$ where $\delta\,\mathbf{E}=\mathbf{E}_1-\mathbf{E}_2$. We obtain

$$\begin{aligned}\nabla\times\delta\,\mathbf{E}&=-j\,\omega\,\mu_c\,\delta\,\mathbf{H}\\ \nabla\times\delta\,\mathbf{H}&=\;\;j\,\omega\,\epsilon_c\,\delta\,\mathbf{E}\end{aligned}\tag{14}$$

The impressed current does not appear in these equations. The difference fields satisfy a source-free form of Maxwell's equations. Applying (8) to these difference difference fields results in

$$\begin{aligned}&\oiint_S \frac{1}{2}\mathrm{Re}(\delta\,\mathbf{E}\times\delta\,\mathbf{H}^*)\cdot\mathbf{ds}\\ &\quad+\frac{1}{2}\iiint_V \sigma_m(\delta\,H)^2\,dv+\frac{1}{2}\iiint_V \sigma\,(\delta\,E)^2\,dv=0\end{aligned}\tag{15}$$

If

$$\oiint_S \frac{1}{2}\mathrm{Re}(\delta\,\mathbf{E}\times\delta\,\mathbf{H}^*)\cdot\mathbf{ds}=0\tag{16}$$

then we must have

$$\frac{1}{2}\iiint_V \sigma_m(\delta\,H)^2\,dv+\frac{1}{2}\iiint_V \sigma\,(\delta\,E)^2\,dv=0\tag{17}$$

If additionally σ,σ_m are positive throughout V then we must have $\delta\,E=\delta\,H=0$ everywhere. That is, $\mathbf{E}_1=\mathbf{E}_2$ and $\mathbf{H}_1=\mathbf{H}_2$ throughout V. This will be the case if

$$(\delta\,\mathbf{E}\times\delta\,\mathbf{H}^*)\cdot\mathbf{ds}=0\tag{18}$$

at every point of S. Now

$$\left|(\delta\,\mathbf{E}\times\delta\,\mathbf{H}^*)\cdot\hat{a}_n\right|=\left|\delta\,\mathbf{E}_t\right|\left|\delta\,\mathbf{H}_t\right|\tag{19}$$

Therefore, if at every point on the surface S we have $\mathbf{E}_{1\mathrm{t}}=\mathbf{E}_{2\mathrm{t}}$ and/or $\mathbf{H}_{1\mathrm{t}}=\mathbf{H}_{2\mathrm{t}}$ then $\mathbf{E}_1=\mathbf{E}_2$ and $\mathbf{H}_1=\mathbf{H}_2$ throughout the volume V. This is the *uniqueness theorem.* We may state it as

> *For given sources $\mathbf{J}_i$ and constitutive parameters within a volume V, the fields are uniquely determined within V if the tangential component of either* **E** *or* **H** *is specified at every point of the bounding surface S.*

Note that the assumption of positive σ,σ_m means that our proof only applies to lossy materials. We can consider a lossless material to be the limit of a lossy material as $\sigma,\sigma_m\to 0$. All real materials have at least some loss (possibly extremely small).

Reciprocity theorem

Suppose two people, A and B, have identical radios. If we know that A has a certain quality reception of B, what can we say about B's reception of A? The *reciprocity theorem* tells us that B will have exactly the same quality reception of A as A has of B. Obviously this result is very important in two-way wireless communication systems. The reciprocity theorem applies to (almost) any "two-port network" in which we swap the locations of transmitter and receiver.

Let the permittivity and permeability throughout a volume V be ϵ_c,μ_c. Suppose when the source is $\mathbf{J}_1$ the fields are $(\mathbf{E}_1,\mathbf{H}_1)$, and when the source is $\mathbf{J}_2$ the fields are $(\mathbf{E}_2,\mathbf{H}_2)$. Therefore

$$\begin{aligned}\nabla\times\mathbf{E}_k&=-j\,\omega\,\mu_c\,\mathbf{H}_k\\ \nabla\times\mathbf{H}_k&=\mathbf{J}_k+j\,\omega\,\epsilon_c\,\mathbf{E}_k\end{aligned}\tag{20}$$

where k is 1 or 2. From Ampere's law we have

$$\begin{aligned}\mathbf{E}_1\cdot\mathbf{J}_2&=\mathbf{E}_1\cdot\nabla\times\mathbf{H}_2-j\,\omega\,\epsilon_c\,\mathbf{E}_1\cdot\mathbf{E}_2\\ \mathbf{E}_2\cdot\mathbf{J}_1&=\mathbf{E}_2\cdot\nabla\times\mathbf{H}_1-j\,\omega\,\epsilon_c\,\mathbf{E}_2\cdot\mathbf{E}_1\end{aligned}\tag{21}$$

Subtracting gives

$$\mathbf{E}_1\cdot\mathbf{J}_2-\mathbf{E}_2\cdot\mathbf{J}_1=\mathbf{E}_1\cdot\nabla\times\mathbf{H}_2-\mathbf{E}_2\cdot\nabla\times\mathbf{H}_1\tag{22}$$

Now

$$\begin{aligned}\nabla\cdot\left(\mathbf{E}_1\times\mathbf{H}_2\right)&=\mathbf{H}_2\cdot\nabla\times\mathbf{E}_1-\mathbf{E}_1\cdot\nabla\times\mathbf{H}_2\\ &=-j\,\omega\,\mu_c\,\mathbf{H}_2\cdot\mathbf{H}_1-\mathbf{E}_1\cdot\nabla\times\mathbf{H}_2\end{aligned}\tag{23}$$

and

$$\begin{aligned}\nabla\cdot\left(\mathbf{E}_2\times\mathbf{H}_1\right)&=\mathbf{H}_1\cdot\nabla\times\mathbf{E}_2-\mathbf{E}_2\cdot\nabla\times\mathbf{H}_1\\ &=-j\,\omega\,\mu_c\,\mathbf{H}_1\cdot\mathbf{H}_2-\mathbf{E}_2\cdot\nabla\times\mathbf{H}_1\end{aligned}\tag{24}$$

Subtracting the first equation from the second results in

$$\nabla\cdot\left(\mathbf{E}_2\times\mathbf{H}_1-\mathbf{E}_1\times\mathbf{H}_2\right)=\mathbf{E}_1\cdot\nabla\times\mathbf{H}_2-\mathbf{E}_2\cdot\nabla\times\mathbf{H}_1\tag{25}$$

Using (22) we have

$$\nabla\cdot\left(\mathbf{E}_2\times\mathbf{H}_1-\mathbf{E}_1\times\mathbf{H}_2\right)=\mathbf{E}_1\cdot\mathbf{J}_2-\mathbf{E}_2\cdot\mathbf{J}_1\tag{26}$$

The divergence theorem allows us to write

$$\oiint_S\left(\mathbf{E}_2\times\mathbf{H}_1-\mathbf{E}_1\times\mathbf{H}_2\right)\cdot\mathbf{dS}=\iiint_V\left(\mathbf{E}_1\cdot\mathbf{J}_2-\mathbf{E}_2\cdot\mathbf{J}_1\right)dV\tag{27}$$

This is one form of the reciprocity theorem. A more useful form results if we let the surface S be a sphere that expands to infinity. If there is at least a small amount of loss then the fields will decay exponentially with increasing radius while the area of the sphere grows only as the radius squared. Therefore the surface integral will vanish and we are left with

$$\boxed{\iiint_V \mathbf{E}_1\cdot\mathbf{J}_2\,dV=\iiint_V \mathbf{E}_2\cdot\mathbf{J}_1\,dV}\tag{28}$$

To see how this applies to the “two radios” situation we mentioned previously, suppose that $\mathbf{J}_1=I\;dl\;\hat{a}_1\,\delta(\mathbf{r}-\mathbf{r}_1)$ and $\mathbf{J}_2=I\;dl\;\hat{a}_2\,\delta(\mathbf{r}-\mathbf{r}_2)$. This is an idealization of two small antennas of length dl located at positions $\mathbf{r}_1,\mathbf{r}_2$, pointing in directions $\hat{a}_1,\hat{a}_2$ and both carrying current I. Doing the integrals gives us

$$I\;dl\;\hat{a}_2\cdot\mathbf{E}_1(\mathbf{r}_2)=I\;dl\;\hat{a}_1\cdot\mathbf{E}_2(\mathbf{r}_1)\tag{29}$$

Now $\hat{a}_2\cdot\mathbf{E}_1(\mathbf{r}_2)$ is the projection of the field due to radio 1, at the location of radio 2, in the direction of the antenna of radio 2. The voltage of the received signal at radio 2 will be proportional to this. We see that the received voltages at the two radios will be the same.

Duality theorem

Duality refers to the symmetry in Maxwell's equations between the electric and magnetic quantities that allows them to be swapped (with some sign changes) and still read as Maxwell's equations. First consider the source-free case in which Maxwell's equations read

$$\begin{aligned} \nabla\times\mathbf{E} &= -j\omega\mu\mathbf{H} \\ \nabla\times\mathbf{H} &= j\omega\epsilon\mathbf{E} \end{aligned} \tag{30}$$

Suppose $\epsilon=f(\mathbf{r})$ and $\mu=g(\mathbf{r})$ describe some environment and you find the solution $(\mathbf{E}_1,\mathbf{H}_1)$ such that

$$\begin{aligned} \nabla\times\mathbf{E}_1 &= -j\omega g(\mathbf{r})\mathbf{H}_1 \\ \nabla\times\mathbf{H}_1 &= j\omega f(\mathbf{r})\mathbf{E}_1 \end{aligned} \tag{31}$$

The "dual problem" would be that in which the permittivity and permeability functions are swapped, namely

$$\begin{aligned} \nabla\times\mathbf{E}_2 &= -j\omega f(\mathbf{r})\mathbf{H}_2 \\ \nabla\times\mathbf{H}_2 &= j\omega g(\mathbf{r})\mathbf{E}_2 \end{aligned} \tag{32}$$

Here $(\mathbf{E}_2,\mathbf{H}_2)$ would be the solution to this new problem. The symmetry of Maxwell's equations allow us to immediate write this solution in terms of our previous solution as

$$\begin{aligned} \mathbf{E}_2 &= -\mathbf{H}_1 \\ \mathbf{H}_2 &= \mathbf{E}_1 \end{aligned} \tag{33}$$

In words, we swap the field vectors and change the sign of the electric field.

Now consider the general case. We will include a fictitious magnetic current and write Maxwell's equations as

$$\begin{aligned} -\nabla\times\mathbf{E} &= \mathbf{M}+j\omega\mu\mathbf{H} \\ \nabla\times\mathbf{H} &= \mathbf{J}+j\omega\epsilon\mathbf{E} \end{aligned} \tag{34}$$

Now, swap the fields, currents and constitutive parameters (switching electric for magnetic and magnetic for electric). This produces

$$\begin{aligned} -\nabla\times\mathbf{H} &= \mathbf{J}+j\omega\epsilon\mathbf{E} \\ \nabla\times\mathbf{E} &= \mathbf{M}+j\omega\mu\mathbf{H} \end{aligned} \tag{35}$$

These are almost Maxwell's equations, but the negative sign is not in the right place. However, if we change the signs of **E** and **J** we have

$$\begin{aligned} -\nabla\times\mathbf{H} &= -\mathbf{J}-j\omega\epsilon\mathbf{E} \\ -\nabla\times\mathbf{E} &= \mathbf{M}+j\omega\mu\mathbf{H} \end{aligned} \tag{36}$$

which is the same as

$$\begin{aligned} -\nabla\times\mathbf{E} &= \mathbf{M}+j\omega\mu\mathbf{H} \\ \nabla\times\mathbf{H} &= \mathbf{J}+j\omega\epsilon\mathbf{E} \end{aligned} \tag{37}$$

and we are back to Maxwell's equations. Here's a summary of duality.

1. Start with a solution to any EM problem.
2. Swap electric and magnetic quantities: fields, currents and constitutive parameters.
3. Change the signs of **E** and **J**.
4. You have a solution to a new EM problem.

Volume equivalence theorem

The idea of a *volume equivalent current* provides a very powerful way to analyze many "scattering" problems. Let the "incident field" be a solution of Maxwell's equation in source-free, free space.

$$\begin{aligned} \nabla\times\mathbf{E}_i &= -j\omega\mu_0\mathbf{H}_i \\ \nabla\times\mathbf{H}_i &= j\omega\epsilon_0\mathbf{E}_i \end{aligned} \tag{38}$$

Now, we introduce some dielectric object described by the relative permittivity function $\epsilon_r(\mathbf{r})$,

$$\begin{aligned} \nabla\times\mathbf{E} &= -j\omega\mu_0\mathbf{H} \\ \nabla\times\mathbf{H} &= j\omega_0\epsilon_0\epsilon_r(\mathbf{r})\mathbf{E} \end{aligned} \tag{39}$$

We are interested in how the object changes the incident field. We write $\mathbf{E}=\mathbf{E}_i+\mathbf{E}_s$ and $\mathbf{H}=\mathbf{H}_i+\mathbf{H}_s$ which expressed the "total field" as the sum of the incident field and the "scattered field" $\mathbf{E}_s,\mathbf{H}_s$. Subtracting the incident field equations from the total field equations gives us

$$\begin{aligned} \nabla\times[\mathbf{E}-\mathbf{E}_i] &= -j\omega\mu_0[\mathbf{H}-\mathbf{H}_i] \\ \nabla\times[\mathbf{H}-\mathbf{H}_i] &= j\omega_0\epsilon_0\epsilon_r(\mathbf{r})\mathbf{E}-j\omega\epsilon_0\mathbf{E}_i \end{aligned} \tag{40}$$

Adding and subtracting $j\omega\epsilon_0\mathbf{E}$ to the right-hand side of the second of these equations puts it in the form

$$\begin{aligned} & j\omega_0\epsilon_0\epsilon_r(\mathbf{r})\mathbf{E}-j\omega\epsilon_0\mathbf{E}_i \\ &\quad = j\omega_0\epsilon_0[\epsilon_r(\mathbf{r})-1]\mathbf{E}+j\omega\epsilon_0[\mathbf{E}-\mathbf{E}_i] \end{aligned} \tag{41}$$

Defining the *equivalent volume current density* as

$$\mathbf{J}_{eq}=j\omega_0\epsilon_0[\epsilon_r(\mathbf{r})-1]\mathbf{E} \tag{42}$$

we then obtain equations for the scattered field

$$\begin{aligned} \nabla\times\mathbf{E}_s &= -j\omega\mu_0\mathbf{H}_s \\ \nabla\times\mathbf{H}_s &= \mathbf{J}_{eq}+j\omega\epsilon_0\mathbf{E}_s \end{aligned} \tag{43}$$

Notice that these correspond to a current $\mathbf{J}_{eq}$ radiating in free space. The effects of the object function $\epsilon_r(\mathbf{r})$ are contained within the equivalent current term. We can immediately write

$$\mathbf{A}_s(\mathbf{r})=\frac{\mu_0}{4\pi}\iiint_V \mathbf{J}_{eq}(\mathbf{r}')\frac{e^{-j\beta_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}dV' \tag{44}$$

and then $\mathbf{E}_s,\mathbf{H}_s$ can be derived from $\mathbf{A}_s$. This is only a formal solution, however, since $\mathbf{J}_{eq}$ depends on the total field **E** and we don't know until we know the scattered field, but we don't know the scattered field until we know $\mathbf{J}_{eq}$. However, this formulation leads to a very powerful approximation in the limit of a "weak scatterer." If $|\epsilon_r(\mathbf{r})-1|\ll 1$ then $|\mathbf{J}_{eq}|$ should be small. It follows that the scattered fields should also be small. If $|\mathbf{E}_s|\ll|\mathbf{E}_i|$ then $\mathbf{J}_{eq}\approx j\,\omega_0\,\epsilon_0[\epsilon_r(\mathbf{r})-1]\mathbf{E}_i$ and

$$\mathbf{A}_s(\mathbf{r})\approx j\,\frac{\omega\mu_0\,\epsilon_0}{4\pi}\iiint_V[\epsilon_r(\mathbf{r}')-1]\,\mathbf{E}_i(\mathbf{r}')\frac{e^{-j\beta_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,dV' \tag{45}$$

depends only on know quantities. This is called the *Born approximation*. One could carry this further and use the scattered field from this approximation to estimate the total field and therefore obtain a better approximation to $\mathbf{J}_{eq}$. From there an improved estimate for $\mathbf{A}_s$ could be obtained.

References

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