Lecture 2c

Potentials

Introduction

In general solving a vector wave equation is a very difficult undertaking, and we would like to avoid working with vector equations whenever possible. In our undergraduate physics courses we saw that an electrostatic vector field can be derived from a scalar potential

$$\mathbf{E}=-\nabla\psi \tag{1}$$

This allows us to work out many electrostatics problems using scalar equations. For example, we can calculate the potential as an integral over the distribution of charge:

$$\psi(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\iiint\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' \tag{2}$$

If we need the vector field $\mathbf{E}$ we can then apply the necessary differential operator to derive it from the scalar potential.

When dealing with non-static fields we cannot, in general, derive the vector fields from a scalar potential. However, we will see that we can derive them from a *vector potential* that can be calculated from the current distribution using a integral relation similar to that above. And there will be special cases where the vector potential effectively reduces to a scalar field.

Magnetic vector potential

Faraday's law $\nabla\times\mathbf{E}=-j\omega\mu\mathbf{H}$ can be rearranged to give

$$\mathbf{H}=\frac{j}{\omega\mu}\nabla\times\mathbf{E} \tag{3}$$

This shows that $\mathbf{H}$ is completely determined by $\mathbf{E}$. For now, let's call $\mathbf{A}=j\mathbf{E}/\omega$. We then have

$$\mathbf{H}=\frac{1}{\mu}\nabla\times\mathbf{A} \tag{4}$$

Therefore $\mathbf{H}$ can be written as the curl of a vector. This is just a restatement of Faraday's law with a redefinition of the electric field times a constant as a new vector $\mathbf{A}$. But here is the "trick." Recall that the curl of the gradient of a scalar field is identically zero, that is $\nabla\times\nabla\psi\equiv 0$. Therefore, we can add the gradient of an arbitrary scalar field into our expression for $\mathbf{A}$ without changing the value of the curl. That is, we can take

$$\boxed{\mathbf{A}=\frac{j}{\omega}(\mathbf{E}+\nabla\psi)} \tag{5}$$

where ψ is *any* scalar field, and (4) will still be true. The scalar field ψ need not have any physical significance; it can be anything we want[1]. It is a mathematical "wiggle term" that, hopefully, we can use to simplify our problems.

The vector $\mathbf{A}$ is called a *vector potential*. We will introduce another vector potential below, so let's be more precise and call $\mathbf{A}$ the *magnetic vector potential* since the magnetic field is obtained from it by a differential operation. The scalar ψ is often called the scalar potential, although it need not have physical significance.

> *The magnetic vector potential is proportional to the electric field plus a "wiggle term."*

Solving for $\mathbf{E}$ we have

$$\mathbf{E}=-j\omega\mathbf{A}-\nabla\psi \tag{6}$$

The addition of the ψ term does not change the value of $\nabla\times\mathbf{A}$ since $\nabla\times\nabla\psi\equiv 0$. The curl of $\mathbf{A}$ is what has physical significance through (4).

The divergence of $\mathbf{A}$ is

$$\nabla\cdot\mathbf{A}=\frac{j}{\omega}\left(\nabla\cdot\mathbf{E}+\nabla^2\psi\right) \tag{7}$$

$\mathbf{E}$ is a physical quantity so $\nabla\cdot\mathbf{E}$ is determined by physics. However, we can take $\nabla^2\psi$ to be an arbitrary function of position. For example, if we set $\nabla^2\psi(\boldsymbol{r})=-q(\boldsymbol{r})/\epsilon_0$ we can interpret $q(\boldsymbol{r})$ as a (fictitious) charge density and $\psi(\boldsymbol{r})$ as the resulting electrostatic field. This is simply Poisson's equation and we know that it has a solution for any function $q(\boldsymbol{r})$, namely (2).

Therefore, we can set the divergence of $\mathbf{A}$ to be an arbitrary function of position. On the other hand, the electric field $\mathbf{E}$ is a physical quantity fully constrained by physics. We can think of the magnetic vector potential $\mathbf{A}$ as a mathematical device, related to $\mathbf{E}$, but with a degree of freedom that will prove very useful for simplifying analysis.

> *The curl of* **A** *is physically constrained but the divergence of* **A** *is arbitrary.*

In general we have

$$\begin{aligned}\mathbf{H}&=\frac{1}{\mu}\nabla\times\mathbf{A}\\ \mathbf{E}&=\frac{1}{j\omega\mu\epsilon}\left(\nabla\times\nabla\times\mathbf{A}-\mu\mathbf{J}\right)\end{aligned} \tag{8}$$

The second equation is from substituting the first equation into Ampere's law. In a source-free region these reduce to

$$\begin{aligned}\mathbf{H}&=\frac{1}{\mu}\nabla\times\mathbf{A}\\ \mathbf{E}&=\frac{1}{j\omega\mu\epsilon}\nabla\times\nabla\times\mathbf{A}\end{aligned} \tag{9}$$

If we take $\mathbf{A}$ to have only a *z* component then we have, for a source-free region

1 It does have to be twice differentiable so that we can take the gradient followed by the curl.

$$\boxed{\begin{aligned} H_x &= \frac{1}{\mu}\frac{\partial}{\partial y}A_z \\ H_y &= -\frac{1}{\mu}\frac{\partial}{\partial x}A_z \\ H_z &= 0 \\ E_x &= -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial x}\frac{\partial}{\partial z}A_z \\ E_y &= -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial y}\frac{\partial}{\partial z}A_z \\ E_z &= \frac{j}{\omega\mu\epsilon}\left[\frac{\partial^2}{\partial x^2}A_z + \frac{\partial^2}{\partial y^2}A_z\right] \end{aligned}} \tag{10}$$

Since **H** has no z component we will refer to this as a TM^z field (the magnetic field is *transverse* to the z direction). It turns out that *any* field that has $H_z \equiv 0$ can be expressed in terms of the scalar field A_z. So, we reduced the problem of finding the five field components E_x, E_y, E_z, H_x, H_y to that of finding the single component A_z. This is a tremendous simplification for analytical purposes.

Helmholtz theorem

The ideas we have been discussing are formalized in the *Helmholtz theorem*. The Helmholtz theorem says that any (physically plausible) vector field **F** can be represented as the combination of the gradient of a scalar field and the curl of a vector field. Mathematically we can write

$$\mathbf{F} = -\nabla\psi + \nabla\times\mathbf{A} \tag{11}$$

We know from Poisson's equation of electrostatics that if

$$\nabla^2\psi(\boldsymbol{r}) = -q(\boldsymbol{r}) \tag{12}$$

then

$$\psi(\boldsymbol{r}) = \frac{1}{4\pi}\iiint\frac{q(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}dV' \tag{13}$$

One thing this tells us is that any scalar field q can be written as the Laplacian of another scalar field ψ, and it even gives us a formula for computing ψ from q.

Now, let **F** be any vector field. Each of the three components of **F** is a scalar field, so we can use our previous result to write $F_x = -\nabla^2 B_x$ where B_x is some other scalar field. We can do likewise for the y and z components of **F**. In vector notation we have

$$\mathbf{F} = -\nabla^2\mathbf{B} \tag{14}$$

This is the vector counterpart to (12). Using a vector identity we write

$$-\nabla^2\mathbf{B} = -\nabla(\nabla\cdot\mathbf{B}) + \nabla\times\nabla\times\mathbf{B} \tag{15}$$

Now, define the scalar and vector fields

$$\begin{aligned} \psi &= \nabla\cdot\mathbf{B} \\ \mathbf{A} &= \nabla\times\mathbf{B} \end{aligned} \tag{16}$$

Then

$$\mathbf{F} = -\nabla\psi + \nabla\times\mathbf{A} \tag{17}$$

This is the Helmholtz theorem. We call ψ the *scalar potential* and **A** the *vector potential*.

What is the advantage of this? Consider the divergence and curl of **F**

$$\begin{aligned} \nabla\cdot\mathbf{F} &= -\nabla^2\psi \\ \nabla\times\mathbf{F} &= \nabla\times\nabla\times\mathbf{A} \end{aligned} \tag{18}$$

This shows that the divergence and curl of a vector field are independent in the sense that they can be separately specified. If the curl is fixed (**A** specified), the divergence can be set to an arbitrary function (specify ψ) and conversely. This will be quite useful to use later on.

Electric vector potential

In a source-free region ($\mathbf{J}=0$) Ampere's law becomes

$$\nabla\times\mathbf{H} = j\omega\epsilon\mathbf{E} \tag{19}$$

We can write

$$\mathbf{E} = -\frac{1}{\epsilon}\nabla\times(j\mathbf{H}/\omega) \tag{20}$$

or

$$\mathbf{E} = -\frac{1}{\epsilon}\nabla\times\mathbf{F} \tag{21}$$

where $\mathbf{F} = j\mathbf{H}/\omega$ is the *electric vector potential*. We can add the gradient of a scalar function to **F** without changing **E**

$$\boxed{\mathbf{F} = \frac{j}{\omega}\left(\mathbf{H} + \nabla\psi_e\right)} \tag{22}$$

Again, this gives us a wiggle term to use for simplifying problems.

Keep in mind that unlike the magnetic vector potential concept, which is always valid,

> *the electric vector potential concept is only applicable in a source-free region.*

When **F** is used we have

$$\begin{aligned} \mathbf{E} &= -\frac{1}{\epsilon}\nabla\times\mathbf{F} \\ \mathbf{H} &= -j\frac{1}{\omega\mu\epsilon}\nabla\times\nabla\times\mathbf{F} \end{aligned} \tag{23}$$

In a source-free region with $\mathbf{F} = \hat{a}_z F_z$

$$
\boxed{\begin{aligned}
E_x &= -\frac{1}{\epsilon}\frac{\partial}{\partial y}F_z \\
E_y &= \frac{1}{\epsilon}\frac{\partial}{\partial x}F_z \\
E_z &= 0 \\
H_x &= -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial x}\frac{\partial}{\partial z}F_z \\
H_y &= -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial y}\frac{\partial}{\partial z}F_z \\
H_z &= \frac{j}{\omega\mu\epsilon}\left[\frac{\partial^2}{\partial x^2}F_z+\frac{\partial^2}{\partial y^2}F_z\right]
\end{aligned}} \tag{24}
$$

Since **E** *has no z* component we will refer to this as a TE^z field (the electric field is *transverse* to the *z* direction). *Any* field in a source-free region that has $E_z \equiv 0$ can be expressed in terms of the scalar field F_z. So, we have reduced the problem of finding the five field components E_x, E_y, H_x, H_y, H_z to that of finding the single component F_z.

Using a combination of A and F

Although we can fully specify both **E** and **H** using the magnetic vector potential **A** alone, or using the electric vector potential **F** alone (in the case of a source-free region), we sometimes find it useful to use **A** and **F** at the same time. This gives us more degrees of freedom that we can use to simplify analysis. When using both **A** and **F** we have

$$
\begin{aligned}
\mathbf{H} &= \frac{1}{\mu}\nabla\times\mathbf{A} - j\frac{1}{\omega\mu\epsilon}\nabla\times\nabla\times\mathbf{F} \\
\mathbf{E} &= -\frac{1}{\epsilon}\nabla\times\mathbf{F} - j\frac{1}{\omega\mu\epsilon}\nabla\times\nabla\times\mathbf{A}
\end{aligned} \tag{25}
$$

As you will show in the homework, an arbitrary plane wave can be represented using just the A_z, F_z components. Just as any (physically plausible) function of time can be represented as a superposition of complex sinusoids (inverse Fourier transform), any (physically plausible) electromagnetic field can be represented as a superposition of plane waves. It follows that any EM field can be represented by specifying A_z, F_z as functions of position. This is a very powerful fact for analytic work.

Hertz vectors

Although we will not use them, an alternate approach is to employ the so-called *Hertz vectors* $\mathbf{\Pi}, \mathbf{\Pi}_m$ These are related to the vector potentials by

$$\mathbf{A} = j\omega\mu\epsilon\mathbf{\Pi} \tag{26}$$

$$\mathbf{F} = j\omega\mu\epsilon\mathbf{\Pi}_m \tag{27}$$

You may see these in the literature and in texts. For example, Ishimaru uses this approach.

Wave equations for A and F

Recall that we started with Faraday's law in our development of the magnetic vector potential concept. So, Faraday's law is already "built in" to our analysis. Let's now consider Ampere's law

$$\nabla\times\mathbf{H} = \mathbf{J} + j\omega\epsilon\mathbf{E} \tag{28}$$

Substituting (4) and (6) results in

$$\nabla\times\left(\frac{1}{\mu}\nabla\times\mathbf{A}\right) = \mathbf{J} + j\omega\epsilon(-j\omega\mathbf{A} - \nabla\psi) \tag{29}$$

Remember that ψ is an arbitrary scalar field, our "wiggle term." Let's assume $\mu = \text{const}$. This allows us to write

$$\nabla\times\nabla\times\mathbf{A} = \mu\mathbf{J} + j\omega\mu\epsilon(-j\omega\mathbf{A} - \nabla\psi) \tag{30}$$

Using a vector identify for $\nabla\times\nabla\times\mathbf{A}$ we have

$$\nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} = \mu\mathbf{J} + j\omega\mu\epsilon(-j\omega\mathbf{A} - \nabla\psi) \tag{31}$$

Rearranging gives us

$$\nabla^2\mathbf{A} + \omega^2\mu\epsilon\mathbf{A} = -\mu\mathbf{J} + \nabla(\nabla\cdot\mathbf{A}) + j\omega\mu\epsilon\nabla\psi \tag{32}$$

The "ugliest" term is $\nabla(\nabla\cdot\mathbf{A})$. If

$$j\omega\mu\epsilon\nabla\psi = -\nabla(\nabla\cdot\mathbf{A}) \tag{33}$$

then this will go away. If both μ and ε are constants then

$$\psi = -\frac{j}{\omega\mu\epsilon}(\nabla\cdot\mathbf{A}) \tag{34}$$

achieves this for us. We have had to assume that both of μ, ϵ are constants, so we are dealing with a simple medium at this point.

We now have the result, valid for any simple medium,

$$\nabla^2\mathbf{A} + \beta^2\mathbf{A} = -\mu\mathbf{J} \tag{35}$$

Compare this to the simple-medium equation we derived for **E**

$$\nabla^2\mathbf{E} + \beta^2\mathbf{E} = j\omega\mu\mathbf{J} + \frac{j}{\omega\epsilon}\nabla(\nabla\cdot\mathbf{J}) \tag{36}$$

Our equation in **A** is much more useful, in particular because each component of **A** depends only on the corresponding component of **J**. For example,

$$\nabla^2 A_z + \beta^2 A_z = -\mu J_z \tag{37}$$

The $\nabla(\nabla\cdot\mathbf{J})$ term in the **E** equation, on the other hand, causes each component of **E** to depend on each component of **J**.

In a source-free simple medium we have

$$
\begin{aligned}
\nabla^2\mathbf{A} + \beta^2\mathbf{A} &= 0 \\
\nabla^2\mathbf{F} + \beta^2\mathbf{F} &= 0
\end{aligned} \tag{38}
$$

The second equation can be derived in a similar manner to the first. We have the interesting result that in a source-free simple medium, **E**, **H**, **A** and **F** all satisfy the Helmholtz equation.

Relation between A and J

Let's go back to (35) and see if we can derive a solution for **A** in terms of **J**. Let's consider only the *z* component of (35) and let J_z be a "point current" at the origin of the form $J_0 \delta(\mathbf{r})$,

$$\nabla^2 A_z + \beta^2 A_z = -\mu J_0 \delta(\mathbf{r}) \tag{39}$$

Notice that if $\beta=0$ then $\nabla^2 A_z = -\mu J_0 \delta(\mathbf{r})$ which looks like Poisson's equation for a point charge at the origin. The solution would be

$$A_z = \frac{\mu}{4\pi} J_0 \frac{1}{r} \tag{40}$$

We will now show that for $\beta \neq 0$ the solution is

$$A_z = \frac{\mu}{4\pi} J_0 \frac{e^{-j\beta r}}{r} \tag{41}$$

The Laplacian for a function or r only (in spherical coordinates) is

$$\nabla^2 A_z = \frac{d^2}{dr^2} A_z + \frac{2}{r} \frac{d}{dr} A_z \tag{42}$$

It is left as an exercise to show that (41) satisfies $\nabla^2 A_z + \beta^2 A_z = 0$ for $r \neq 0$. We then need to show that $\nabla^2 A_z + \beta^2 A_z$ behaves like $-\mu J_0 \delta(\mathbf{r})$ in the sense that

$$\iiint_V (\nabla^2 A_z + \beta^2 A_z) dV = -\mu J_0 \tag{43}$$

over any volume *V* containing the origin. Consider a sphere of radius $r_0 \to 0$. In spherical coordinates we have $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$.

$$\begin{aligned}\iiint_V A_z \, dV &= \frac{\mu}{4\pi} J_0 \iiint_V \frac{e^{-j\beta r}}{r} r^2 \sin\theta \, dr \, d\theta \, d\phi \\ &\to 0\end{aligned} \tag{44}$$

since $r_0 \to 0$ and the integrand has a factor of *r*. Therefore the $\beta^2 A_z$ term of (43) does not contribute and we must have

$$\iiint_V \nabla^2 A_z \, dV = -\mu J_0 \tag{45}$$

The divergence theorem gives us

$$\iiint_V \nabla^2 A_z \, dV = \oiint_S \nabla A_z \cdot \mathbf{dS} \tag{46}$$

Using $\nabla A_z = \hat{a}_r \frac{d}{dr} A_z$ we obtain

$$\nabla A_z = \frac{-\mu J_0}{4\pi} e^{-j\beta r} \left(\frac{1 + j\beta r}{r^2} \right) \hat{a}_r \tag{47}$$

Then

$$\begin{aligned}\oiint_S \nabla A_z \cdot \mathbf{dS} &= \frac{-\mu J_0}{4\pi} e^{-j\beta r_0} \left(\frac{1 + j\beta r_0}{r_0^2} \right) 4\pi r_0^2 \\ &\to -\mu J_0\end{aligned} \tag{48}$$

So, we have shown that

$$A_z = \frac{\mu}{4\pi} J_0 \frac{e^{-j\beta r}}{r} \tag{49}$$

is the solution to

$$\nabla^2 A_z + \beta^2 A_z = -\mu J_0 \delta(\mathbf{r}) \tag{50}$$

Now let J_z be any scalar field. We can write

$$J_z(\mathbf{r}) = \iiint J_z(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') dV \tag{51}$$

The potential produced by $J_z(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}')$ will be

$$\frac{\mu}{4\pi} J_z(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{52}$$

Therefore

$$A_z(\mathbf{r}) = \frac{\mu}{4\pi} \iiint_V J_z(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dV' \tag{53}$$

We could repeat exactly the same steps for the *x* and *y* coordinates. The result is the vector equation

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \iiint_V \mathbf{J}(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dV'} \tag{54}$$

This is a very powerful result. We cannot derive such a simple relationship between **J** and either **E** or **H**. This tells us that if, for example, **J** has only an *x* component, then **A** will have only an *x* component. This would apply to a wire antenna parallel to the *x* axis. All six components of **E** and **H** may be derived from the single component of **A**.

References

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2. Balanis, C. A., *Advanced Engineering Electromagnetics*, Wiley, 1989, ISBN 0-471-62194-3.