Lecture 2b

Constitutive parameters and boundary conditions

Permittivity

In this section we are going to use a very simplistic model to develop the concept of the permittivity of a material. This development is not based on rigorous atomic physics, so it can't be expected to rigorously describe real materials. However it will give a plausibility argument for why the permittivity of real materials vary, show frequency dependence and exhibit internal loss.

In free space Gauss's law is $\epsilon_0 \nabla\cdot\mathbf{E}=q$. Using the divergence theorem we obtain the integral form

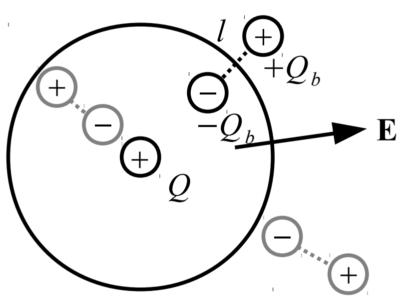
$$\epsilon_0 \oiint_S \mathbf{E}\cdot\mathbf{dS}=Q_{\text{enc}}=\iiint_V q\,dV \tag{1}$$

Let the surface be a sphere of radius r and let the charge be distributed with spherical symmetry. Then on the surface of the sphere

$$\mathbf{E}=\frac{Q_{\text{enc}}}{4\pi\epsilon_0 r^2}\hat{a}_r \tag{2}$$

where Q_{enc} is the total charge inside the sphere.

Imagine a point charge Q placed at the origin and embedded in some material. The atoms of the material consist of equal amounts of positive and negative charge. Assume the positive charge (nucleus) is fixed in place. The electric field due to Q will tend move these negative "bound" charges. Some of those charge will cross the surface of the sphere, change the value of Q_{enc} .



Treating the binding forces within the atom as an equivalent spring, the displacement l of the negative charge is determined by the equilibrium between the electric force and the restoring force

$$k\,l=E\,Q_b \tag{3}$$

where k is the "spring constant" of the atom. Let N be the number of atoms per unit volume. Any atom within a distance l of the sphere's surface will put a negative charge into the sphere. There are

$$N\,4\pi r^2 l=N\,4\pi\,r^2\,E\,Q_b/k \tag{4}$$

such atoms. The total charge within the sphere is therefore

$$Q_{\text{enc}}=Q-Q_b\,N\,4\pi\,r^2\,E\,Q_b/k \tag{5}$$

The electric field amplitude is

$$\begin{aligned} E&=\frac{Q-Q_b N\,4\pi r^2 E\,Q_b/k}{4\pi\epsilon_0 r^2} \\ &=\frac{Q}{4\pi\epsilon_0 r^2}-\frac{Q_b^2 N}{k\,\epsilon_0}E \end{aligned} \tag{6}$$

Solving for $\mathbf{E}$ we have

$$\mathbf{E}=\frac{Q}{4\pi\epsilon_0(1+\chi_e)r^2}\hat{a}_r \tag{7}$$

where, in our model, $\chi_e=Q_b^2 N/k\epsilon_0$ is the *electric susceptibility* of the material. It is dimensionless. The effects of the bound charges are accounted for by this constant and our expression for $\mathbf{E}$ only contains the "free" charge Q explicitly.

We see that when the susceptibility is non-zero, the electric field is weakened relative to the value it would have in free space. This is the *dielectric* effect. We are led to define the *permittivity* of the material as

$$\boxed{\epsilon=\epsilon_0(1+\chi_e)=\epsilon_0\epsilon_r} \tag{8}$$

The constant $\epsilon_r=1+\chi_e$ is called the *dielectric constant* or the *relative permittivity*. It is dimensionless. In this simple model, different materials will have different permittivities if they have different parameters Q_b, N, k .

If the electric flux density is given by $\mathbf{D}=\epsilon\,\mathbf{E}$ then

$$\mathbf{D}=\frac{Q}{4\pi r^2}\hat{a}_r \tag{9}$$

This is the same result we would get for a charge Q in free space. By allowing the permittivity to be a function of the material, that is, by allowing the relation between **D** and **E** to vary from material to material, we are freed from having to explicitly account for bound charges. We can write Gauss's law as $\nabla\cdot\mathbf{D}=q$ where q refers only to free charge and $\mathbf{D}=\epsilon\,\mathbf{E}$. The constant ϵ accounts for the effect of bound charges.

Frequency dependence

Our previous model was a static one. If the field is oscillating then any displaced charges will be oscillating also. Let's see what effect this might have. Let the electric field and charge displacement be functions of time $E(t), l(t)$. In addition to the spring force $k\,l(t)$ there will be the inertial force ma or $m\,d^2 l(t)/dt^2$ where m is the mass of the charge. There may also be a "friction" force proportional to velocity $\alpha\,v(t)$ or $\alpha\,d\,l(t)/dt$. Therefore we have the equation

$$m\frac{d^2}{dt^2}l(t)+\alpha\frac{d}{dt}l(t)+k\,l(t)=Q_b E(t) \qquad (10)$$

Assume all time dependence is given by $e^{j\omega t}$ and let l and E be phasors. Then

$$-\omega^2 m\,l+j\,\omega\,\alpha\,l+k\,l=Q_b E \qquad (11)$$

Solving for l we have

$$l=\frac{Q_b E}{k[(1-\omega^2/\omega_0^2)+j\,\omega/\omega_d]} \qquad (12)$$

where $\omega_0^2=k/m$ and $\omega_d=k/\alpha$. Comparing this to the static result $l=E\,Q_b/k$ suggests that the susceptibility would be a function of frequency given by

$$\chi_e=\frac{\chi_{e0}}{(1-\omega^2/\omega_0^2)+j\,\omega/\omega_d} \qquad (13)$$

where $\chi_{e0}=Q_b^2 N/k\epsilon_0$ is the static value. Multiplying by the conjugate of the denominator gives us

$$\chi_e=\chi_{e0}\frac{(1-\omega^2/\omega_0^2)-j\,\omega/\omega_d}{(1-\omega^2/\omega_0^2)^2+(\omega/\omega_d)^2} \qquad (14)$$

This shows that χ_e should be in general complex and have a negative imaginary part. At $\omega=\omega_0$, χ_e becomes purely imaginary. This is an example of a "resonance" frequency. Near a resonance frequency the material will strongly absorb power from the field.

Recall that the dot product of force and velocity gives the power transferred to a particle. Force is proportional to E and velocity is the time derivative of l, or $j\,\omega\,l$ in phasor notation. Therefore $1/2\,\mathrm{Re}\{j\,\omega\,l\,E\}$ is proportional to the time average power transferred from the field to the charge. Taking E to be real, if l is real then the power is zero. From (12) we see that l will have an imaginary part only if the ω_d term is present. This is the case if α is non-zero. Recall from undergraduate physics that a spring dissipates energy when it has a "damping" term of this form. We see that χ_e is complex if there is an internal loss ("damping") mechanism in the material.

If the susceptibility is complex, the same will be true for the permittivity $\epsilon=\epsilon_0(1+\chi_e)$. When we want to emphasize the complex nature of the permittivity we will often use the notation

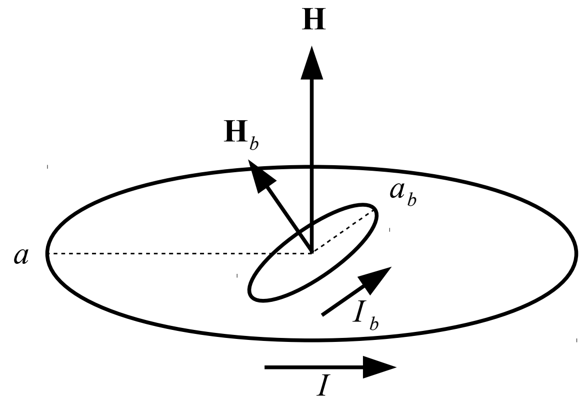
$$\epsilon_c=\epsilon'-j\,\epsilon'' \qquad (15)$$

As we've seen in a previous lecture, conductivity can be represented by $\epsilon''=\sigma/\omega$. Whether ϵ'' is due to electrical conductivity or loss mechanisms within atoms (the "friction force" term) it will result in a "lossy" material in which power is transferred from the field to the material (typically manifesting itself as heat). This can useful, as in microwave heating, or a problem, as in absorption losses in fiber optics.

We mentioned that our analysis is not to be considered rigorous. However, real materials do more-or-less display the type of behavior we have sketched out. The permittivity is generally a function of frequency and displays at some level a loss mechanism, that is, the permittivity is complex. Typically, however, there are several resonance frequencies.

Permeability

The atoms in certain materials, most notably ferromagnetic materials, behave as though they have a net circulation of current that creates a small magnetic field. In an unmagnetized state these numerous magnetic dipoles are randomly oriented and the resulting net magnetic field is zero. If an external magnetic field is applied, however, the dipoles will tend to align with it thereby increasing the magnetic field.



Consider the situation illustrated above. Here I_b represents a "bound" current moving around a loop of radius a_b within an atom, and I is an external current in a loop of radius a. In your undergraduate EM course you worked out $H=I/(2\mathrm{a})$ as the magnetic field produced by I at the center of the loop. From $\mathbf{F}=Q\,\mathbf{v}\times\mathbf{B}$ the force on the loop tends to cause its magnetic field $H_b=I_b/(2a_b)$ to align with $\mathbf{H}$. The total magnetic field at the origin is then $\mathbf{H}+\mathbf{H}_b$ and $\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{H}_b)$.

More realistically there are numerous bound currents, and the magnetic field **H** tends to cause these to more-or-less align with **H**. The stronger **H** is the more they align. The net result is that the total $\mathbf{H}_b$ increases with increasing **H**. We write $\mathbf{H}_b=\chi_m\mathbf{H}$ where χ_m is the *magnetic susceptibility*. We then have $\mathbf{B}=\mu_0(1+\chi_m)\mathbf{H}=\mu\,\mathbf{H}$ where the permeability of the medium is given by

$$\boxed{\mu=\mu_0(1+\chi_m)=\mu_0\,\mu_r} \qquad (16)$$

The advantage of this point of view is that **H** is directly related to "free" currents while the effects of "bound" currents are accounted for by μ .

If the field **H** is oscillating with frequency ω, then generally

$$\mu_c=\mu'-j\mu'' \qquad (17)$$

and the permeability is complex and a function of frequency.

While the great majority of materials show some dielectric effect, magnetic effects (permeability) are typically significant only for ferromagnetic materials such as iron and cobalt and some rare-earth elements.

Types of media

Simple media

In a simple medium ϵ, μ are (possibly complex) scalar constants. If both ϵ, μ are real then the medium is *lossless*. If either are complex then the medium is *lossy*. We will mostly be concerned with simple media in this course.

Dispersive media

If either of ϵ, μ are a functions of frequency (true at some level for all materials) then the medium is *dispersive*. The time-harmonic (phasor) approach we are using easily accommodates dispersion. We will examine the effects of dispersion in a later lecture.

Inhomogeneous media

If $\epsilon=\epsilon(\mathbf{r})$ and/or $\mu=\mu(\mathbf{r})$ then the medium is *inhomogeneous*. Otherwise it is *homogeneous*. A simple medium is homogeneous, but a homogeneous medium is not necessarily simple – it might be anisotropic, for example. If inhomogeneities are piece-wise constant it is easier to analyze a problem as a collection of simple media with appropriate boundary conditions. This is the approach we will take for most of this course. If the inhomogeneities are described by continuous functions then the problem is generally much more difficult. We will briefly consider the simplest case of a one-dimensional inhomogeneous medium in a future lecture.

Anisotropic media

We have seen how the permittivity of a material is related to the displacement of bound charges within it. In an *anisotropic* medium charges are more easily displaced in certain directions. This means that the permittivity can be different for different directions of **E**, i.e., it is polarization dependent. In such media **D** and **E** are not necessarily parallel and the permittivity is described by a three-by-three matrix

$$\epsilon=\begin{pmatrix}\epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz}\\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz}\\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz}\end{pmatrix} \tag{18}$$

so that

$$\begin{aligned} D_x &= \epsilon_{xx}E_x+\epsilon_{xy}E_y+\epsilon_{xz}E_z\\ D_y &= \epsilon_{yx}E_x+\epsilon_{yy}E_y+\epsilon_{yz}E_z\\ D_z &= \epsilon_{zx}E_x+\epsilon_{zy}E_y+\epsilon_{zz}E_z \end{aligned} \tag{19}$$

and each component of **D** depends on all the components of **E**.

Anisotropic materials have important applications. Anisotropic permittivity is particularly useful at optical frequencies with LCD displays being a prime example. The Yariv and Yeh reference is a good source for more information.

The permeability could also be anisotropic in which case μ would be a three-by-three matrix. Anisotropic magnetic materials have important applications in radio circuits such as "RF circulators" that allow wireless communication systems to separate transmitted and received signals present in a common antenna.

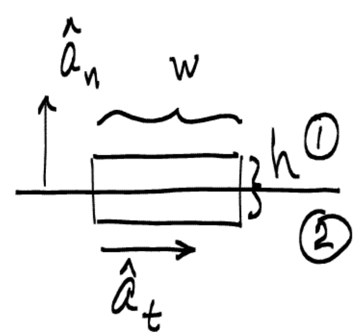
Non-linear media

In a non-linear medium the relations between **D** and **E** and/or between **B** and **H** depend on the field strength. Equivalently the permittivity and/or permeability are functions of the fields $\epsilon=\epsilon(\mathbf{E})$, $\mu=\mu(\mathbf{H})$. In a non-linear system the principle of superposition does not apply. If a linear material is driven with a certain frequency its response is at the same frequency, hence the power of the phasor method. An interesting effect of a non-linear material is that it can create field components at frequencies other than the driving frequency.

Boundary conditions

At an interface between two simple media we need to consider the *boundary conditions* on the field vectors. Consider the illustration below.

Here $\hat{a}_n$ is normal to the interface between medium 1 and medium 2 and $\hat{a}_t$ is any unit vector tangent to the interface. Calculating $\oint \mathbf{E}\cdot\mathbf{dl}$ around the very small rectangular loop and letting $h\to 0$, we have $\oint \mathbf{E}\cdot\mathbf{dl}=w\,\mathbf{E}_2\cdot\hat{a}_t-w\,\mathbf{E}_1\cdot\hat{a}_t$. Since



$\oint \mathbf{E}\cdot\mathbf{dl}=-j\omega\iint_S \mu\,\mathbf{H}\cdot\mathbf{dS}$ and the area of of the rectangular surface is zero as $h\to 0$, we have, provided **H** is finite, $\mathbf{E}_2\cdot\hat{a}_t=\mathbf{E}_1\cdot\hat{a}_t$. This is true for any tangent vector $\hat{a}_t$. Therefore the tangential components of **E** are the same in the two media.

We also have

$$\oint \mathbf{H}\cdot\mathbf{dl}=\iint_S \mathbf{J}\cdot\mathbf{dS}+j\omega\iint_S \epsilon\,\mathbf{E}\cdot\mathbf{dS} \tag{20}$$

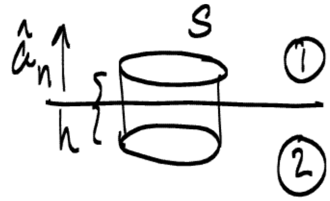
So, provided **E** and **J** are finite, $\mathbf{H}_2\cdot\hat{a}_t=\mathbf{H}_1\cdot\hat{a}_t$. Summarizing:

$$\boxed{\begin{array}{c}\mathbf{E}_{1t}=\mathbf{E}_{2t}\\ \mathbf{H}_{1t}=\mathbf{H}_{2t}\end{array}} \tag{21}$$

The tangential components of **E** and **H** are continuous across a boundary. This can also be expressed as

$$\begin{array}{c}\hat{a}_n\times\mathbf{E}_1=\hat{a}_n\times\mathbf{E}_2\\ \hat{a}_n\times\mathbf{H}_1=\hat{a}_n\times\mathbf{H}_2\end{array} \tag{22}$$

Now consider a cylinder of height $h\rightarrow 0$ as shown below. From $\nabla\cdot(\mu\mathbf{H})=0$ we have $\oint\!\!\oint(\mu\mathbf{H})\cdot\mathbf{ds}=0$. This reduces to $(\mu_1\mathbf{H}_1)\cdot\hat{a}_n=(\mu_2\mathbf{H}_2)\cdot\hat{a}_n$. From $\nabla\cdot(\epsilon\mathbf{E})=q$ we have $\oint\!\!\oint(\epsilon\mathbf{E})\cdot\mathbf{ds}=Q$ where Q is the charge within the cylinder. If q is finite then $Q\rightarrow 0$ as $h\rightarrow 0$ and we have $(\epsilon_1\mathbf{E}_1)\cdot\hat{a}_n=(\epsilon_2\mathbf{E}_2)\cdot\hat{a}_n$.



Summarizing:

$$\boxed{\begin{array}{c}\hat{a}_n\cdot(\epsilon_1\mathbf{E}_1)=\hat{a}_n\cdot(\epsilon_2\mathbf{E}_2)\\ \hat{a}_n\cdot(\mu_1\mathbf{H}_1)=\hat{a}_n\cdot(\mu_2\mathbf{H}_2)\end{array}} \tag{23}$$

The normal components of $\epsilon\mathbf{E}$ and $\mu\mathbf{H}$ are continuous across a boundary.

Special case of a PEC

In deriving boundary conditions we have assumed that all parameters and field quantities are finite. An exception would be the ideal model of a *perfect electric conductor* (PEC) in which $\sigma\rightarrow\infty$ and $\mathbf{J}=\sigma\mathbf{E}$ could be infinite even for a finite **E**. Suppose that in the previous Figures medium 2 is a PEC. $\mathbf{E}_2$ must be zero or else infinite current would flow. From Faraday's law we have that **H** must be zero also. Therefore $\mathbf{E}_{1t}=0$ at the boundary. The same cannot be said for $\mathbf{H}_{1t}$, however, because $\iint_S\mathbf{J}\cdot\mathbf{dS}$ will not necessarily go to zero as $h\rightarrow 0$, since $\sigma\rightarrow\infty$. In the homework we will consider the problem starting with medium 2 having a finite conductivity and then letting $\sigma\rightarrow\infty$. We will find that the current in medium 2 is localized on its surface resulting in a *surface current density* $\mathbf{J}_s$ having units of $\mathrm{A/m}$. The boundary conditions when medium 2 is PEC are

$$\boxed{\begin{array}{r}\mathbf{E}_{1t}=0\\ \hat{a}_n\times\mathbf{H}_{1t}=\mathbf{J}_s\end{array}} \tag{24}$$

Similarly, at the surface of a PEC q may become infinite (finite amount of charge in a region of zero thickness) and $\oint\!\!\oint(\epsilon\mathbf{E})\cdot\mathbf{ds}=Q$ can be finite even as the surface shrinks to zero height. The result is

$$\boxed{\begin{array}{c}\hat{a}_n\cdot(\epsilon_1\mathbf{E}_1)=q_s\\ \hat{a}_n\cdot(\mu_1\mathbf{H}_1)=0\end{array}} \tag{25}$$

where q_s is the *surface charge density* in $\mathrm{C/m^2}$. In practice we will see that we can use $\mathbf{E}_{1t}=0$ as the boundary condition and $\hat{a}_n\times\mathbf{H}_{1t}=\mathbf{J}_s$ then allows us to compute the surface current on the PEC.

References

1. Yariv and Yeh, *Optical Waves in Crystals: Propagation and Control of Laser Radiation*, Wiley, 2002, ISBN-13: 978-0471430810.