Lecture 2a

Field vectors and Maxwell's equations

Electric charge and the electric field

Electromagnetics is concerned with the interaction of electric charges at rest and in motion. Maxwell described the classic demonstration of electrical phenomena as follows:

> "Let a piece of glass and a piece of resin ... be rubbed together ... they will now attract each other."

Moreover, if we do the same with a second piece of glass and resin,

> "... two pieces of glass repel each other ... each piece of glass attracts each piece of resin .. two pieces of resin repel each other."

Unlike the gravitational force, which is always attractive, the electric force can be attractive or repulsive. This shows that there are two types of electric charge. Like charges repel and different charges attract.

> "It is the established practice of men of science to call the vitreous [glass] electrification positive, and the resinous [resin] electrification negative ... [this is] a matter of arbitrary convention ..."

We now know that positive electric charge resides in protons and negative electric charge resides in electrons. These charges are equal in magnitude and opposite in sign. We refer to this magnitude as the *elementary charge*. It is usually denoted by e (which is, unfortunately, easily confused with the e of e^x). The unit of electric charge is the *coulomb* (C). One coulomb corresponds to $6.24150962915265 \cdot 10^{18}$ elementary charges. If an object has a net positive charge then there are fewer electrons than protons. If it has a net negative charge then there are more electrons than protons.

If a charge Q is placed at rest at some point P and it experiences a force **F**, then the *electric field* at the point P is defined via the equation

$$\mathbf{F} = Q\,\mathbf{E} \tag{1}$$

The electric field therefore has units of force per charge $\mathrm{N/C}$. We can write $1\,\mathrm{Nm/Cm} = 1\,\mathrm{V/m}$ where one *volt* is $1\,\mathrm{V} = 1\,\mathrm{J/C}$. The electric field is almost always expressed in units of *volts per meter*.

Coulomb's experiment demonstrated that the magnitude of the force between two charges Q_1, Q_2 in free space is given by

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 r^2} \tag{2}$$

where r is the distance between the charges. Here ϵ_0 is a constant called the *permittivity* of free space. The units of permittivity are $\mathrm{C^2/Nm^2} = \mathrm{C^2/Jm}$. Since $1\,\mathrm{J} = 1\,\mathrm{VC}$ this is $\mathrm{C/Vm}$. Defining the derived unit *farad* as $1\,\mathrm{F} = 1\,\mathrm{C/V}$, the units of permittivity are $\mathrm{F/m}$. The free-space value is $\epsilon_0 = 8.8542 \cdot 10^{-12}\,\mathrm{F/m}$.

Coulomb's experiment shows that the electric field due to a charge Q at the origin is (expressed in spherical coordinates)

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r \tag{3}$$

It is convenient to define the *electric flux density* in free space as $\mathbf{D} = \epsilon_0 \mathbf{E}$ then

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{a}_r \tag{4}$$

There is a more direct relationship between **D** and Q since the constant ϵ_0 is not present. The units of electric flux density are $\mathrm{C/m^2} = \mathrm{As/m^2}$. Using the divergence theorem, the relation between **D** and Q can expressed as

$$\nabla \cdot \mathbf{D} = q \tag{5}$$

where $q = dQ/dV$ is the volume charge density. This is the differential form of *Gauss's law*. It is one of the four Maxwell's equations.

Electric current and the magnetic field

Another type of force acts on charges only when they are in motion. If a charge Q has velocity **v** and experiences a force **F**, then the *magnetic flux density* **B** is defined via the equation

$$\mathbf{F} = Q\,\mathbf{v} \times \mathbf{B} \tag{6}$$

The units of **B** are $\mathrm{Ns/Cm}$ or $\mathrm{Vs/m^2}$. If we define the derived unit *weber* (Wb) as $1\,\mathrm{Wb} = 1\,\mathrm{Vs}$, then the units of **B** are $\mathrm{Wb/m^2}$.

We refer to moving charges as electric *current*. One *ampere* of current corresponds to one coulomb of charge per second flowing through a reference surface. The experiments of Ampere showed that a current produces a magnetic field which in turn produces a force on other currents. In fact the modern definition of the ampere is

> *The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2 x 10^{-7} newton per meter of length.*

(http://physics.nist.gov/cuu/Units/ampere.html.) The coulomb is then defined as 1 ampere second, that is, the amount of charge flowing in one second through a conductor carrying a current of one ampere. A point of interest is that it has been proposed that the coulomb be *defined* as exactly $6.24150962915265 \cdot 10^{18}$ elementary charges. This, together with the definition of the ampere, the meter and the second would fix the value of the kilogram as a derived unit. The kilogram would no longer have to be defined as the mass of a particular chunk of metal siting in Paris.

Ampere's experiment showed the following. If a wire carries a current I along the z axis, it creates a magnetic flux density in free space (expressed in cylindrical coordinates) of

$$\mathbf{B}=\frac{\mu_0 I}{2\pi r}\hat{a}_\phi \quad (7)$$

Here μ_0 is a constant called the *permeability* of free space. It has units of $\mathrm{Wb/Am}$. Defining the derived unit *henry* as $1\,\mathrm{H}=1\,\mathrm{Wb/A}$, the units of permeability are $\mathrm{H/m}$. The free-space value is $\mu_0=4\pi\cdot 10^{-7}\,\mathrm{H/m}$.

It is convenient to define the *magnetic field* in free space **H** as $\mathbf{H}=(1/\mu_0)\mathbf{B}$ (we usually write this as $\mathbf{B}=\mu_0\mathbf{H}$).Then

$$\mathbf{H}=\frac{I}{2\pi r}\hat{a}_\phi \quad (8)$$

There is a more direct relationship between I and **H** since the constant μ_0 is not present. The units of **H** are $\mathrm{A/m}$. Using Stoke's theorem the relation between I and **H** can be expressed as

$$\nabla\times\mathbf{H}=\mathbf{J} \quad (9)$$

where $\mathbf{J}=\hat{a}_J\, dI/dS$ is the electric *current density*. It has units of $\mathrm{A/m^2}$. Here dI is the current flowing through a surface of area dS normal to $\hat{a}_J$, the unit vector in the direction of charge velocity. This is the (static) differential form of *Ampere's law*. It is the basic equation of *magnetostatics*.

Lorentz force law

If both electric and magnetic forces act on a charge, we have

$$\boxed{\mathbf{F}=Q(\mathbf{E}+\mathbf{v}\times\mathbf{B})} \quad (10)$$

This is called the *Lorentz force law*. We can take it as the operational definition of the electric and magnetic fields. It shows how the fields manifest themselves through forces on charges. Maxwell's equations tell us what kinds of field distributions are possible, how they interact with each other and how charges at rest or in motion create fields.

The fact that we have two different vectors associated with each of the electric and magnetic fields can be confusing. Let's review this. The vectors **E** and **B** are usually considered the fundamental field vectors and are most closely related to the way fields exert forces on charges and currents. The vectors **D** and **H** are usually considered auxiliary vectors. They are most closely related to the way in which charges and currents produce fields.

In free space there is nothing physically profound about this. For example, **D** and **E** can be considered as two versions of the same field vector. The difference in magnitude and units is simply due to the way we define charge, current and force. Indeed we could redefine the unit of charge so that Coulomb's law read $F=Q_1 Q_2/r^2$.

The situation will become more complex in materials. If we keep track of *all* charges and currents, including those "bound" in the atoms that make up the material, then we can continue to use $\mathbf{D}=\epsilon_0\mathbf{E}$ and $\mathbf{B}=\mu_0\mathbf{H}$. However, this is a nearly impossible task. Instead, it is much easier to keep track only of "free charge" and "free current," those charges and currents that are not bound within atoms, and to incorporate the effects of "bound charge" and "bound current" into the relations between **D**,**E** and **B**,**H**. This will lead to expressions $\mathbf{D}=\epsilon\mathbf{E}$ and $\mathbf{B}=\mu\mathbf{H}$ where ϵ,μ are now functions of the material. We will consider this in more detail in a future lecture.

Maxwell's equations

Faraday's law

Electrostatic fields are conservative. The total work done on a charge in moving around a closed loop is zero, or $\oint\mathbf{E}\cdot d\mathbf{l}=0$. In differential form

$$\nabla\times\mathbf{E}=0 \quad (11)$$

(Conservative fields are "irrotational.")

The experiments of Faraday showed that a time-changing magnetic field produces an electric field that is generally not conservative. In fact the time-domain version of Faraday's law states

$$\nabla\times\mathbf{E}=-\partial_t\mathbf{B} \quad (12)$$

In general $\oint\mathbf{E}\cdot d\mathbf{l}\neq 0$ and net work can be done on a charge around a closed loop. This is the basis of, among other things, most electric power generation. In phasor notation we have

$$\nabla\times\mathbf{E}=-j\omega\mathbf{B} \quad (13)$$

Using $\mathbf{B}=\mu\mathbf{H}$ this becomes

$$\nabla\times\mathbf{E}=-j\omega\mu\mathbf{H} \quad (14)$$

This is the differential form of Faraday's law that we will primarily use. Using Stoke's theorem we obtain the integral form

$$\oint_L \mathbf{E}\cdot d\mathbf{l}=-j\omega\iint_S(\mu\mathbf{H})\cdot d\mathbf{s} \quad (15)$$

If we take the divergence of both sides of the differential form and use $\nabla\cdot(\nabla\times\mathbf{E})=0$ we find

$$\nabla\cdot(\mu\mathbf{H})=0 \quad (16)$$

This is usually expressed as $\nabla\cdot\mathbf{B}=0$ and is considered one of the four Maxwell's equations. However, it is implicit in Faraday's law and we will not make much use of it directly. By analogy with $\nabla\cdot\mathbf{D}=q$ we see that magnetic charge does not exist.

Ampere's law

Ampere's law for static fields is

$$\nabla\times\mathbf{H}=\mathbf{J} \tag{17}$$

Since $\nabla\cdot\nabla\times\mathbf{H}=0$ this requires $\nabla\cdot\mathbf{J}=0$. By the divergence theorem the total current flowing through any closed surface would be zero. This is true in the static case, but in general there is no reason we cannot have a net current flow across a closed surface for some finite period of time. Maxwell modified Ampere's law to apply to the time-varying case by adding a "displacement current" term $\partial\mathbf{D}/\partial t$ to get

$$\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial}{\partial t}\mathbf{D} \tag{18}$$

In the phasor domain this becomes

$$\nabla\times\mathbf{H}=\mathbf{J}+j\omega\mathbf{D} \tag{19}$$

Using $\mathbf{D}=\epsilon\,\mathbf{E}$ we obtain

$$\nabla\times\mathbf{H}=\mathbf{J}+j\omega\epsilon\mathbf{E} \tag{20}$$

This is the form of Ampere's law we will most often employ. Using Stoke's theorem we obtain the integral form

$$\oint_L \mathbf{H}\cdot\mathbf{dl}=\iint_S \mathbf{J}\cdot\mathbf{ds}+j\omega\iint_S(\epsilon\mathbf{E})\cdot\mathbf{ds} \tag{21}$$

Since $\nabla\cdot(\nabla\times\boldsymbol{H})=0$, taking the divergence of Ampere's law results in

$$\nabla\cdot\mathbf{J}=-j\omega\nabla\cdot(\epsilon\mathbf{E}) \tag{22}$$

From the divergence theorem, $\nabla\cdot\mathbf{J}\,dV$ is the net current flowing out of the infinitesimal volume dV. If charge is conserved, then charge leaving the volume will result in a decrease in the charge remaining in the volume. Let the charge within the volume be $q\,dV$ where q is the volume charge density. Then the conservation of charge gives $\nabla\cdot\mathbf{J}=-dq/dt$ or $\nabla\cdot\mathbf{J}=-j\omega q$ in the phasor domain. We see that Maxwell's modified version of Ampere's law together with the conservation of charge requires

$$-j\omega q=\nabla\cdot\mathbf{J}=-j\omega\nabla\cdot(\epsilon\mathbf{E}) \tag{23}$$

and therefore

$$\nabla\cdot(\epsilon\mathbf{E})=q \tag{24}$$

This is the differential form of Gauss's law. Thus Gauss's law in implicit in Ampere's law. A useful way to restate the conservation of charge is

$$q=\frac{j}{\omega}\nabla\cdot\mathbf{J} \tag{25}$$

Here then are the form of Maxwell's equations that we will employ in this course:

$$\boxed{\begin{aligned}\nabla\times\mathbf{E}&=-j\omega\mu\mathbf{H}\\ \nabla\times\mathbf{H}&=\mathbf{J}+j\omega\epsilon\mathbf{E}\end{aligned}} \tag{26}$$

The two divergence equations $\nabla\cdot(\mu\,\mathbf{H})=0$ and $\nabla\cdot(\epsilon\,\mathbf{E})=q$ are implicitly contained within these curl relations.

Electric current and "magnetic current"

It is often quite convenient to consider the current density as composed of two contributions: *impressed current* and *conduction current*. We write

$$\mathbf{J}=\mathbf{J}_i+\mathbf{J}_c \tag{27}$$

We think of impressed current as that which is forced to have a certain value regardless of the fields and is therefore known *a priori*. An example would be the current that is forced to flow in a wire antenna. Conduction current is that which flows in response to the fields. For linear materials the conduction current is given by the differential form of Ohm's law:

$$\mathbf{J}_c=\sigma\,\mathbf{E} \tag{28}$$

Here σ is the *conductivity* of the material. It has units of $\mathrm{A/Vm}$ or $\mathrm{S/m}$ where the *siemen* is defined as $1\,\mathrm{S}=1\,\mathrm{A/V}$.

Ampere's law becomes

$$\nabla\times\mathbf{H}=\mathbf{J}_i+\sigma\,\mathbf{E}+j\omega\epsilon\mathbf{E} \tag{29}$$

We can use the following bookkeeping notation

$$\begin{aligned}\sigma\,\mathbf{E}+j\omega\epsilon\,\mathbf{E}&=j\omega\left(\epsilon-j\frac{\sigma}{\omega}\right)\mathbf{E}\\ &=j\omega\epsilon_c\,\mathbf{E}\end{aligned} \tag{30}$$

where the *complex permittivity* is $\epsilon_c=\epsilon-j\sigma/\omega$. We can use either of the following notations to represent complex permittivity

$$\epsilon_c=\epsilon'-j\epsilon''=\epsilon-j\sigma/\omega \tag{31}$$

In practice the dielectric properties of a material are specified by giving ϵ',ϵ'' at various frequencies. We can then define the effective conductivity to be

$$\sigma=\omega\epsilon'' \tag{32}$$

There need not be any connection between this effective conductivity and the DC conductivity of the material.

Likewise, the permeability can have an imaginary part. By analogy with permittivity, we can treat this as an equivalent "magnetic conductivity"

$$\mu_c=\mu'-j\mu''=\mu-j\sigma_m/\omega \tag{33}$$

even though there is no such thing as a DC magnetic conductivity because there is no such thing as magnetic charge and magnetic current. Instead, μ'' is due to loss mechanisms within the atomic structure of a material. Nonetheless, the effect is the same as that which would be produced by a (fictitious) magnetic current. We will call the magnetic current **M** and write

$$\mathbf{M}=\sigma_m\mathbf{H}=\omega\mu''\mathbf{H} \tag{34}$$

Faraday's law can then be written.

$$\nabla\times\mathbf{E}=-\mathbf{M}-j\omega\mu\,\mathbf{H} \tag{35}$$

In a future lecture we will consider ways to determine the ϵ',ϵ'' and μ',μ'' values of a material from reflection and transmission measurements.

Current and power

Current consists of charges in motion. If the charges are subjected a force then that force does work on the charges as they move through some displacement.

Current density **J** has units of $\mathrm{A/m^2}$. Let q be a charge density ($\mathrm{C/m^3}$) and **v** be a velocity ($\mathrm{m/s}$). Then q**v** has units of $\mathrm{C/sm^2}$ or $\mathrm{A/m^2}$. Indeed $\mathbf{J}=q\,\mathbf{v}$ - current density is charge density times charge velocity.

Now consider a charge $Q=q\,dV$ moving with velocity **v** and let it be acted on by an electric field **E** (**v** and **E** are phasors). This creates a force Q**E**. The dot product of force and velocity is work done per unit time, or power. The time-average power is therefore $W=(1/2)\,q\,dV\,\mathrm{Re}\left(\mathbf{E}\cdot\mathbf{v}^*\right)$. Since $\mathbf{J}=q\,\mathbf{v}$, the time-average power density $w=dW/dV$ ($\mathrm{W/m^3}$) is

$$w=\frac{1}{2}\mathrm{Re}\left(\mathbf{E}\cdot\mathbf{J}^*\right) \tag{36}$$

This is the power transferred from the field to the current per unit volume. If this is negative then the current is doing work on the field. This is the case in a transmitting antenna. If **J** is given by Ohm's law then

$$w=\frac{1}{2}\sigma\,E^2 \tag{37}$$

This is never negative, so power is always lost from the field in a conductive medium.

Wave Equations

Maxwell's equations give a complete description of electromagnetic phenomena. They are, however, coupled first-order partial differential equations in the two vector unknowns **E** and **H**. When pursuing analytic solutions it is typically more useful to derive a single equation in a single unknown, say **E**. We will refer to an equation of this sort as a *wave equation*.

Let's start with Faraday's law

$$\nabla\times\mathbf{E}=-j\,\omega\mu\,\mathbf{H} \tag{38}$$

We would like to express the **H** factor in terms of **E**. What we have available is Ampere's law that gives $\nabla\times\mathbf{H}$ in terms of **E** and **J**. If we assume μ is a constant then we can take the curl of both sides of Faraday's law to get

$$\nabla\times\nabla\times\mathbf{E}=-j\,\omega\mu\,\nabla\times\mathbf{H} \tag{39}$$

(If μ was not constant there would be a $\nabla\mu\times\mathbf{H}$ term to deal with.) Now we can substitute for $\nabla\times\mathbf{H}$ from Ampere's law to get

$$\nabla\times\nabla\times\mathbf{E}=-j\,\omega\mu\left(\mathbf{J}+j\,\omega\,\epsilon\mathbf{E}\right) \tag{40}$$

Rearranging a bit, we have

$$-\nabla\times\nabla\times\mathbf{E}+\omega^2\mu\,\epsilon\,\mathbf{E}=j\omega\mu\,\mathbf{J} \tag{41}$$

This is a wave equation for **E** when $\mu=\mathrm{const}$ but ϵ is arbitrary. The source (forcing) function is proportional to the current density **J**. It is not a very useful equation, however, because $\nabla\times\nabla\times\mathbf{E}$ is such a complicated differential form. So, let's try to simplify this.

Since $\nabla\times\nabla\times\mathbf{E}=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}$ we can write

$$\nabla^2\mathbf{E}+\omega^2\mu\,\epsilon\,\mathbf{E}=j\,\omega\mu\,\mathbf{J}+\nabla(\nabla\cdot\mathbf{E}) \tag{42}$$

This looks a bit better. Let's try to simplify some more. If the medium is simple (both μ,ϵ are constant) then

$$\nabla\cdot(\epsilon\,\mathbf{E})=\epsilon\,\nabla\cdot\mathbf{E}=q \tag{43}$$

Therefore

$$\nabla(\nabla\cdot\mathbf{E})=\frac{1}{\epsilon}\nabla q \tag{44}$$

and

$$\nabla^2\mathbf{E}+\beta^2\mathbf{E}=j\omega\mu\,\mathbf{J}+\frac{1}{\epsilon}\nabla q \tag{45}$$

where we have defined the constant

$$\boxed{\beta^2=\omega^2\mu\,\epsilon} \tag{46}$$

Finally, using $q=\dfrac{j}{\omega}\nabla\cdot\mathbf{J}$ we can write

$$\nabla^2\mathbf{E}+\beta^2\mathbf{E}=j\omega\mu\,\mathbf{J}+\frac{j}{\omega\,\epsilon}\nabla(\nabla\cdot\mathbf{J}) \tag{47}$$

This is a wave equation for **E** in any simple medium with arbitrary current density **J**. It still is not a very useful equation due to the $\nabla(\nabla\cdot\mathbf{J})$ term.

If the medium is source-free ($\mathbf{J}=0$) then the right-hand side is zero and we are left with

$$\nabla^2\mathbf{E}+\beta^2\mathbf{E}=0 \tag{48}$$

This is the (homogeneous) *Helmholtz equation*. By similar steps one can show that in a source-free simple medium **H** also satisfies the Helmholtz equation.

$$\nabla^2\mathbf{H}+\beta^2\,\mathbf{H}=0 \tag{49}$$

The Helmholtz equation is reasonably tractable and will be the basis of most of our analysis in this course.

References

1. Maxwell, J. C., *A Treatise on Electricity and Magnetism*, Dover, 1954 (reprint of 1891 edition), ISBN 0-486-60636-

8 (vol. 1) and 0-486-60637-6 (vol. 2).

2. Balanis, C. A., *Advanced Engineering Electromagnetics*, Wiley, 1989, ISBN 0-471-62194-3.

Appendix: units

Fundamental units: meter (m), kilogram (kg), second (s), ampere (A).

newton (force)	$1\,\text{N} = 1\,\text{kg m/s}^2$
joule (energy)	$1\,\text{J} = 1\,\text{Nm}$
watt (power	$1\,\text{W} = 1\,\text{J/s}$
coulomb (charge)	$1\,\text{C} = 1\,\text{As}$
volt (electric potential)	$1\,\text{V} = 1\,\text{J/C}$
farad (capacitance)	$1\,\text{F} = 1\,\text{C/V}$
weber (magnetic flux)	$1\,\text{Wb} = 1\,\text{Vs}$
henry (inductance)	$1\,\text{H} = 1\,\text{Wb/A}$
ohm (resistance)	$1\,\Omega = 1\,\text{A/V}$
siemen (conductivity)	$1\,\text{S} = 1\,\text{V/A}$

Appendix: constants

Elementary charge

$$e = 1.602 \cdot 10^{-19}\,\text{C}$$

Permeability of free space

$$\mu_0 \equiv 4\pi \cdot 10^{-7}\,\text{H/m}$$

Speed of light in free space (exact by definition)

$$c \equiv 299{,}792{,}458\,\text{m/s}$$

Permittivity of free space

$$\epsilon_0 \equiv \frac{1}{\mu_0 c^2} = 8.8542 \cdot 10^{-12}\,\text{F/m}$$