Lecture 1d

Boundary-value problems

Introduction

The material in this lecture might appear a bit abstract, and many electromagnetic texts don't emphasize it. However, it is very fundamental in understanding the nature of the solutions to Maxwell's equations that we will derive.

We will see that electromagnetic systems are typically described by a partial differential equation (PDE). Solving a PDE is not easy. The only systematic approach we will have is to try a *separation of variables* where we hope that a solution can be found in the factored form $E(u,v,w)=f(u)g(v)h(w)$. If successful this turns a PDE of three variables into three ODEs that can then be solved by the methods we have discussed previously.

Here's the issue. Say our "trick" works and we find one or more solutions of the form $f(u)g(v)h(w)$, which we will call a *mode* of the EM system. This is great as far as it goes, but these are just a subset of an infinite number of possible solutions. Many of those solutions can't be written as $f(u)g(v)h(w)$. How can we "get at" the most general solution to Maxwell's equations in our particular geometry?

Sturm-Liouville theory provides the answer. Using it we will be able to show that *any* solution of the systems we will study can be represented as a linear combination of the factored modes we were fortunate to be able to find. The idea is analogous to Fourier series which you have seen allows any function to be represented as a series of variations of one type of fundamental function (sinusoids). We will need to extend this concept to non-sinusoidal functions and to three dimensions.

Boundary value problems

In our review of differential equation theory we have noted that we can always find a unique solution to a 2nd order HLODE satisfying the two initial conditions $y(x_0)=y_0$, $y'(x_0)=y'_0$. In EM applications, however, we typically have *boundary conditions* of the form $y(x_1)=y_1$, $y(x_2)=y_2$ rather than initial conditions. Initial value problems always have a unique solution. Boundary value problems may or may not have a solution. Typically solutions exist only when certain parameters of the equation take on one of a set of values.

As an example, let's consider a simple boundary value problem consisting of the 2nd order HLODE

$$y''+\Omega^2 y=0 \quad (1)$$

with the boundary conditions

$$y(0)=y(1)=0 \quad (2)$$

The general solution to the HLODE is

$$y(x)=a_1\cos(\Omega x)+a_2\sin(\Omega x) \quad (3)$$

Since $\cos(0)=1$, the boundary condition (BC) $y(0)=0$ requires $a_1=0$, and

$$y(x)=a_2\sin(\Omega x) \quad (4)$$

The second BC then requires

$$a_2\sin(\Omega)=0\rightarrow\Omega=k\pi\ ,\ k=1,2,3,\ldots \quad (5)$$

Of course we could also have taken $a_2=0$ but this gives the trivial solution $y(x)\equiv 0$. We see that our boundary value problem has a solution only when the parameter Ω takes on one of the values *eigenvalues* $\pi, 2\pi, 3\pi,\ldots$ giving *eigenfunction* solutions of the form

$$y(x)=a_k\sin(k\pi x) \quad (6)$$

where a_k is some constant. In EM problems solutions of this type will represent *modes* of the EM field in some structure. Now, consider a sum of all of our "modes" for the current problem:

$$f(x)=\sum_{k=1}^{\infty} a_k\sin(k\pi x) \quad (7)$$

This is a *Fourier series* that can be used to represent *any* function $f(x)$ over the interval $0<x<1$. The coefficients are given by

$$a_k=2\int_0^1 f(x)\sin(k\pi x)\,dx \quad (8)$$

As you may recall from undergraduate courses, the usefulness of Fourier series rests on two properties. First, the functions $\sin(k\pi x)$ are *orthogonal* to each other. This allows us to calculate the coefficients via an integral of the form (8). Second, the functions $\sin(k\pi x)$ are *complete* over the interval $(0,1)$, meaning that the Fourier series can represent *any* continuous function $f(x)$ over that interval.

Of course, most boundary value problems will not have the functions $\sin(k\pi x)$ as solutions, so the modes we obtain will not form a basis for a Fourier series. However, we will see that the modes of an arbitrary boundary value problem typically form a complete, orthogonal set of functions that can be used as the basis of a *generalized Fourier series* that can be just as useful as a "regular" Fourier series.

Sturm-Liouville problem

Consider the following 2nd order HLODE

$$[u(x)y']'+[\lambda w(x)+v(x)]y=0 \quad (9)$$

over the interval $x_1\le x\le x_2$ with homogeneous boundary conditions

$$\begin{aligned} a_1 y(x_1)+b_1 y'(x_1)=0 \ \ or \ \ u(x_1)=0 \\ a_2 y(x_2)+b_2 y'(x_2)=0 \ \ or \ \ u(x_1)=0 \end{aligned} \quad (10)$$

Here λ is a parameter to be determined in the solution of the problem. We assume that $u(x), v(x), w(x)$ and $u'(x)$ are real and continuous over the interval. This is the called the *Sturm-Liouville problem*. Expanding the Sturm-Liouville form and dividing through by $u(x)$ we obtain

$$y''+\frac{u'(x)}{u(x)}y'+\frac{[v(x)+\lambda w(x)]}{u(x)}y=0 \qquad (11)$$

This has the standard form of a 2nd order HLODE

$$y''+p(x)y'+q(x)y=0 \qquad (12)$$

with

$$\begin{aligned} p(x)&=\frac{u'(x)}{u(x)} \\ q(x)&=\frac{\lambda w(x)+v(x)}{u(x)} \end{aligned} \qquad (13)$$

We can invert these relations to obtain

$$\begin{aligned} u(x)&=u(0)e^{\int_0^x p(x)dx} \\ v(x)+\lambda w(x)&=q(x)u(x) \end{aligned} \qquad (14)$$

These equations allows us to put any 2nd order HLODE into the Sturm-Liouville form. For example, consider the *Legendre equation*

$$y''-\frac{2x}{1-x^2}y'+\frac{n(n+1)}{1-x^2}y=0 \qquad (15)$$

with $-1\le x\le 1$. By inspection we can identify $u(x)=1-x^2$, $u'(x)=-2x$, $v(x)=0$, $w(x)=1$ and $\lambda=n(n+1)$.

In general, solutions to the boundary value problem are possible only for certain values of the parameter λ which we call the *eigenvalues* of the problem.

$$\lambda=\lambda_0,\lambda_1,\ldots \qquad (16)$$

Let's call $y_k(x)$ the solution corresponding to $\lambda=\lambda_k$. These are the *eigenfunctions* of the problem.

Orthogonality property

Consider any two of the eigenfunctions, $y_m(x), y_n(x)$. We have

$$\begin{aligned} [u\,y_m']'+v\,y_m&=-\lambda_m w\,y_m \\ [u\,y_n']'+v\,y_n&=-\lambda_n w\,y_n \end{aligned} \qquad (17)$$

Multiplying the first equation by y_n, the second by y_m and subtracting we get

$$y_n[u\,y_m']'-y_m[u\,y_n']'=(\lambda_n-\lambda_m)w\,y_m y_n \qquad (18)$$

We are going to integrate both sides. Let's work on the left side first. Recall "integration by parts"

$$\int a\,db=ab-\int b\,da \qquad (19)$$

Taking $a=y_n$ and $db=[u\,y_m']'\,dx$, we have

$$\int y_n[u\,y_m']'dx=y_n[u\,y_m']-\int u\,y_m'\,y_n'\,dx \qquad (20)$$

Likewise we have

$$\int y_m[u\,y_n']'dx=y_m[u\,y_n']-\int u\,y_n'\,y_m'\,dx \qquad (21)$$

Subtracting (21) from (20) and putting in the limits of integration gives us

$$\int_{x_1}^{x_2}\left(y_n[u\,y_m']'-y_m[u\,y_n']'\right)dx=u[y_n y_m'-y_m y_n']\Big|_{x_1}^{x_2} \qquad (22)$$

At $x=x_1$ the right-hand side is zero. This is because either $u(x_1)=0$ or using the BC's we have

$$\begin{aligned} y_n(x_1)\,y_m(x_1)'&=y_n(x_1)\left[-\frac{a_1}{b_1}y_m(x_1)\right] \\ &=-\frac{a_1}{b_1}y_n(x_1)\,y_m(x_1) \end{aligned} \qquad (23)$$

We get the same result for $y_m(x_1)\,y_n(x_1)'$. Therefore

$$y_n(x_1)\,y_m'(x_1)-y_m(x_1)\,y_n'(x_1)=0 \qquad (24)$$

The same is true at x_2. Therefore

$$u[y_n y_m'-y_m y_n']\Big|_{x_1}^{x_2}=0 \qquad (25)$$

so

$$\int_{x_1}^{x_2}\left(y_n[u\,y_m']'-y_m[u\,y_n']'\right)dx=0 \qquad (26)$$

and it follows that

$$(\lambda_n-\lambda_m)\int_{x_1}^{x_2} w(x)\,y_m(x)\,y_n(x)dx=0 \qquad (27)$$

Since the two eigenfunctions correspond to different eigenvalues we have $(\lambda_n-\lambda_m)\neq 0$ and finally

$$\int_{x_1}^{x_2} w(x)\,y_m(x)\,y_n(x)dx=0 \qquad (28)$$

This shows that $y_m(x), y_n(x)$ are orthogonal over the interval $[x_1,x_2]$ with *weighting function* $w(x)$.

A somewhat similar type of argument shows that if $w(x)\ge 0$ over the interval $[x_1,x_2]$ then all the eigenvalues λ_k and eigenfunctions $y_k(x)$ are real.

Oscillatory behavior of eigenfunctions

With $u=u(x)$ and $g=\lambda w(x)+v(x)$ (9) becomes

$$[u\,y']'+g\,y=0 \qquad (29)$$

If u and g were positive constants the solution would be sinusoidal with radian frequency $\Omega=\sqrt{g/u}$. The number of zeros of that function over an interval of length L would be $L\Omega/\pi$. For the general case, *Sturm's comparison theorem* says that if $u_{max}=max[u(x)]$ and $g_{min}=min[\lambda w(x)+v(x)]$

are positive then the number of zeros of the solution of (9) over an interval of length L is bounded by

$$\#\ zeros\ \geq \frac{L}{\pi}\sqrt{\frac{g_{min}}{u_{max}}} \tag{30}$$

If $w(x)$ is positive then $g_{min}=min[\lambda w(x)+v(x)]$ will increase as λ increases and the solution $y(x)$ will oscillate more rapidly (have more zeros) with increasing λ.

Arguing in this manner, it can be proven that if $u(x)>0$ and $w(x)>0$ over the interval $x_1<x<x_2$ then the eigenvalues of (9) form an infinite sequence

$$\lambda_0<\lambda_1<\cdots<\lambda_n<\cdots \tag{31}$$

which is unbounded, that is, $\lambda_n\to\infty$ as $n\to\infty$. Moreover if $y_n(x)$ is the eigenfunction corresponding to eigenvalue λ_n, then $y_n(x)$ has precisely n zeros over $x_1<x<x_2$. In this sense the solutions to a Sturm-Liouville problem having $u, w>0$ are qualitatively analogous to sinusoids.

Generalized Fourier series

In your undergraduate coursework you have seen how useful it can be to represent arbitrary functions as Fourier series constructed from sines and cosines.

Let's take an arbitrary linear combination of eigenfunctions of a general Sturm-Liouville problem

$$y(x)=\sum_{k=0}^{\infty} a_k y_k(x) \tag{32}$$

Since each of the $y_k(x)$ satisfies the boundary conditions of the problem, $y(x)$ will also. Multiplying both sides by $w(x)y_m(x)$ and integrating over the interval $[x_1, x_2]$ allows us to solve for the coefficient a_m because of the orthogonality of the eigenfunctions

$$a_m=\frac{\int_{x_1}^{x_2} w(x)\,y_m(x)\,y(x)\,dx}{\int_{x_1}^{x_2} w(x)\,y_m^2(x)\,dx} \tag{33}$$

This gives us a *generalized Fourier series* that can be used to represent any piece-wise continuous function over the interval (x_1, x_2).

For systems with $u(x)>0$ and $w(x)>0$ over (x_1, x_2), the unboundedness of the eigenvalues and the oscillatory behavior of the eigenfunctions allow one to prove that the eigenfunctions form a *complete set*, that is, any (reasonably well-behaved) function can be rigorously represented by its generalized Fourier series.

Summary

A 2nd order LODE in Sturm-Liouville form

$$[u(x)y']'+[\lambda w(x)+v(x)]y=0 \tag{34}$$

over the closed interval $x_1\leq x\leq x_2$ with boundary conditions

$$\begin{aligned} a_1 y(x_1)+b_1 y'(x_1)=0 \;\; or \;\; u(x_1)=0 \\ a_2 y(x_2)+b_2 y'(x_2)=0 \;\; or \;\; u(x_1)=0 \end{aligned} \tag{35}$$

and under the conditions that $u(x), v(x), w(x)$ and $u'(x)$ are real and continuous and $u(x)>0, w(x)>0$ over the open interval $x_1<x<x_2$ has an infinite, unbounded set of eigenvalues

$$\lambda_0<\lambda_1<\cdots<\lambda_n<\cdots \tag{36}$$

The corresponding eigenfunctions form a complete set that can be used to represent any function $y(x)$ over $x_1<x<x_2$ as the generalized Fourier series

$$y(x)=\sum_{k=0}^{\infty} a_k y_k(x) \tag{37}$$

The coefficients are given by

$$a_m=\frac{\int_{x_1}^{x_2} w(x)\,y_m(x)\,y(x)\,dx}{\int_{x_1}^{x_2} w(x)\,y_m^2(x)\,dx} \tag{38}$$

Delta functions

On a different, but related topic, we will make use of the *impulse* or *delta function* in this course. You learned about the delta function $\delta(t)$ in your undergraduate linear-systems courses. Its usefulness primarily stems from the following facts. Any function $x(t)$ can be represented as the convolution of itself with the delta function: $x(t)=x(t)*\delta(t)$. If $\delta(t)$ is input to a linear, time-invariant system, the output will be the impulse response $h(t)$. If $x(t)$ is input to the same system the output will be the convolution of the input and the impulse response: $y(t)=x(t)*h(t)$. Thus in a sense, if we can determine how the system responds to a delta function we know everything there is to know about the system. What will be different in this course is that we will primarily be concerned with delta functions in space rather than in time.

The one-dimensional delta function can be defined as

$$\delta(x)=\lim_{w\to 0}\begin{cases} 1/w & |x|\leq w/2 \\ 0 & |x|>w/2 \end{cases} \tag{39}$$

The delta function has the property that it is zero except when its argument is zero (where it's infinite) and

$$\int \delta(x)\,dx=1 \tag{40}$$

The sampling property of the delta function

$$\int f(x)\,\delta(x-x_0)\,dx=f(x_0) \tag{41}$$

is very useful. We can represent a three-dimensional delta function as a product of one-dimensional delta functions

$$\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z) \tag{42}$$

Or we could define it as a spherical function

$$\delta(\mathbf{r}) = \lim_{w \to 0} \begin{cases} 3/(4\pi w^3) & 0 \le r \le w \\ 0 & r > w \end{cases} \tag{43}$$

In either case

$$\iiint \delta(\mathbf{r}) dv = 1 \tag{44}$$

and the three-dimensional sampling property is

$$\iiint f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0) dv = f(\mathbf{r}_0) \tag{45}$$

Like other functions, delta functions can be expanded over the eigenfunctions of a Sturm-Liouville system. For example

$$\delta(x-\xi) = \sum_{k=0}^{\infty} a_k y_k(x) \tag{46}$$

where

$$a_m = \frac{w(\xi) y_m(\xi)}{\int_{x_1}^{x_2} w(x) y_m^2(x) dx} \tag{47}$$

gives the coefficients.

References

1. Kreyszig, E., *Advanced Engineering Mathematics*, 4th Ed.,Wiley, 1979, ISBN 0-471-020140-7. Section 4.8: “Sturm-Liouville Problem” gives proofs of orthogonality of the eigenfunctions and realness of the eigenvalues.

2. Ince, E. L., *Ordinary Differential Equations*, Dover, 1956, ISBN 0-486-60349-0. Chapter X: “The Sturmian Theory and Its Later Developments” gives proofs of everything we have discussed here.