Lecture 1c

Linear ordinary differential equations

Terminology

A general 1st order, *linear ordinary differential equation* (LODE) can be expressed in the form

$$y' + p(x)y = r(x) \tag{1}$$

If the *forcing function* $r(x)$ is identically zero, then the LODE is *homogeneous* (HLODE) and has the form

$$y' + p(x)y = 0 \tag{2}$$

A general 2nd order LODE can be expressed in the form

$$y'' + p(x)y' + q(x)y = r(x) \tag{3}$$

If the forcing function is identically zero, we have a 2nd order HLODE

$$y'' + p(x)y' + q(x)y = 0 \tag{4}$$

The solution of 2nd order HLODEs will be a central issue for us.

Numerical solutions

Although in this course we are primarily interested in analytic solutions, it is instructive to briefly consider how one might develop numerical solutions to a differential equation. Suppose we limit consider to a discrete set of *x* values

$$x_k = x_0 + k\,h \tag{5}$$

where *h* is a constant step size, and the corresponding function values are denoted

$$y_k = y(x_k) \tag{6}$$

From the definition of the derivative

$$y'(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} \tag{7}$$

we have the first derivative approximation

$$y'_k \approx \frac{y_{k+1} - y_k}{h} \tag{8}$$

and the second derivative approximation

$$y''_k \approx \frac{\dfrac{y_{k+2} - y_{k+1}}{h} - \dfrac{y_{k+1} - y_k}{h}}{h} = \frac{y_{k+2} - 2y_{k+1} + y_k}{h^2} \tag{9}$$

Equation (2) becomes

$$y_{k+1} \approx (1 - h\,p_k)\,y_k \tag{10}$$

where $p_k = p(x_k)$. We see that 1st order LODE is simply a formula for computing a future value of the function, y_{k+1} , from the present value y_k . From this expression we see that if $y(x) = 0$ for any value of *x* then $y(x) \equiv 0$ for all values of *x*. That is, a non-trivial solution of a 1st order HLODE can never vanish[1]. To initialize this solution we need a single *initial condition*, for example, $y(x_0) = y_0$. We can take $y(x_0) = 1$ to define a specific function $y_1(x)$. Any solution to the HLODE can then be written $y(x) = a_1 y_1(x)$.

Our approximation to (4) reads

$$y_{k+2} \approx 2y_{k+1} - y_k - h^2\left[q_k y_k + p_k(y_{k+1} - y_k)/h\right] \tag{11}$$

or

$$y_{k+2} \approx y_{k+1} + (1 - hp_k)(y_{k+1} - y_k) - h^2 q_k y_k \tag{12}$$

We see that a 2nd order HLODE is a formula for using two function values, y_k, y_{k+1} , to compute the future value y_{k+2} . To initialize this solution we need *two initial conditions*, say $y(x_0) = y_0$ and $y(x_1) = y_1$. Equivalently we could specify $y(x_0) = y_0$ and $y'(x_0) = (y_1 - y_0)/h$. We can form two specific solutions by, say, taking $y(x_0) = 1$ and $y'(x_0) = 0$ to generate a solution $y_1(x)$ and taking $y(x_0) = 0$ and $y'(x_0) = 1$ to generate a solution $y_2(x)$. Any solution to the HLODE can then be written as $y(x) = a_1 y_1(x) + a_2 y_2(x)$. The initial conditions will be $y(x_0) = a_1$ and $y'(x_0) = a_2$.

For the 2nd order HLODE, it is possible for a non-trivial solution to vanish at some value of *x*. If, say, $x_k = 0$ but $x_{k+1} \neq 0$ then in general $y_{k+2} \neq 0$. A solution is identically zero if and only if both the function and the 1st derivative vanish at some value of *x*.

Analytic solutions

The solution to the 1st order HLODE is

$$y(x) = y_0\, e^{-\int_{x_0}^{x} p(s)\,ds} \tag{13}$$

The constant y_0 is determined by the initial condition $y(x_0) = y_0$.

For the 2nd order HLODE, there always exists two *linearly independent* solutions $y_1(x)$ and $y_2(x)$. By linearly independent we mean that $a_1 y_1(x) + a_2 y_2(x) \equiv 0$ if and only if $a_1 = a_2 = 0$. If the functions are *linearly dependent*, then $a_1 y_1(x) \equiv -a_2 y_2(x)$ and they are the same function to within a multiplicative constant. Any solution to the homogeneous equation can be written as a linear combination of two linearly independent solutions

$$y(x) = a_1 y_1(x) + a_2 y_2(x) \tag{14}$$

The two constants are uniquely determined by initial conditions of the form

1 This assumes that $p(x)$ remains finite.

$$y(x_0)=b_1 \\ y'(x_0)=b_2 \tag{15}$$

We have

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{16}$$

The determinant of the matrix is called the *Wronskian*

$$W(x)=y_1(x)y_2'(x)-y_1'(x)y_2(x) \tag{17}$$

If $W(x_0)\neq 0$ then the matrix in (16) is non-singular (hence invertible) and our problem has a unique solution. Let's take a derivative of $W(x)$. We have

$$\begin{aligned} W'(x) &= y_1(x)y_2''(x)-y_1''(x)y_2(x) \\ &= y_1[-p\,y_2'-q\,y_2]-y_2[-p\,y_1'-q\,y_1] \\ &= -p(x)[y_1(x)y_2'(x)-y_1'(x)y_2(x)] \\ &= -p(x)W(x) \end{aligned} \tag{18}$$

Therefore the Wronskian satisfies

$$W'(x)+p(x)W(x)=0 \tag{19}$$

This 1st order HLODE has solution

$$W(x)=W(x_0)e^{-\int_{x_0}^{x} p(t)\,dt} \tag{20}$$

If the Wronskian is nonzero at any point, then it never vanishes in a domain in which $p(x)$ is finite, and the initial value problem will have a unique solution.

Constant coefficients

If the functions $p(x), q(x)$ are constants, our HLODE has the form

$$y''+p\,y'+q\,y=0 \tag{21}$$

In this case there are two solutions of the form $y_1=e^{s_1 x}$ and $y_2=e^{s_2 x}$ where s_1, s_2 are the (possibly complex) roots of

$$s^2+p\,s+q=0 \tag{22}$$

namely

$$s=\frac{-p\pm\sqrt{p^2-4q}}{2} \tag{23}$$

For $s_1\neq s_2$, $y_1/y_2=e^{(s_1-s_2)x}$ is not a constant, so the two solutions are linearly independent. This fails only when $p^2=4q$, since this results in $s_1=s_2=-p/2$. In this case it is easy to verify that two linearly independent solutions are $y_1=e^{-px/2}$ and $y_2=x\,e^{-px/2}$.

Non-constant coefficients

The general 2nd order LODE is

$$y''+p(x)y'+q(x)y=r(x) \tag{24}$$

If the functions $p(x), q(x), r(x)$ are *analytic* they can be represented by convergent power series

$$p(x)=\sum_{n=0}^{\infty} p_n x^n \\ q(x)=\sum_{n=0}^{\infty} q_n x^n \\ r(x)=\sum_{n=0}^{\infty} r_n x^n \tag{25}$$

where p_n, q_n, r_n are constants. The solutions are also analytic and can be represented by convergent power series. That is, we can write

$$y(x)=\sum_{n=0}^{\infty} a_n x^n \tag{26}$$

Note that

$$y(0)=a_0 \\ y'(0)=a_1 \tag{27}$$

So the first two coefficients are determined by initial conditions at $x=0$. For the derivatives we have

$$y'(x)=\sum_{n=1}^{\infty} n\,a_n x^{n-1} \tag{28}$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \tag{29}$$

Substituting these into the LODE, along with the power series for $p(x), q(x), r(x)$, we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+\sum_{n=0}^{\infty} p_n x^n \sum_{n=1}^{\infty} n\,a_n x^{n-1} \\ +\sum_{n=0}^{\infty} q_n x^n \sum_{n=0}^{\infty} a_n x^n=\sum_{n=0}^{\infty} r_n x^n \tag{30}$$

By collecting the coefficient of x^n on the left-hand side and equating it to r_n we can solve for a_n. In general we will obtain a recursion relation for the coefficients $a_2, a_3, a_4, \dots$ in terms of a_0, a_1. If a_0, a_1 have been determined by initial conditions, then this is the solution to our particular initial value problem. If we want to develop two linearly independent solutions we could take, for example, $a_0=1, a_1=0$ for $y_1(x)$ and $a_0=0, a_1=1$ for $y_2(x)$.

Frobenius method

If in a 2nd order HLODE the functions $p(x), q(x)$ are not analytic, it may still be possible to find series solutions. Typically for us the problem is that either p or q has a *singularity* at $x=0$, for example, $p(x)=1/x$ "blows up" at $x=0$. If $x\,p(x)$ and $x^2 q(x)$ are analytic then the singularity is a *regular singularity*; otherwise it is an irregular singularity. For a regular singularity, there is at least one solution of the form[2]

$$y(x)=x^r\sum_{n=0}^{\infty} a_n x^n \tag{31}$$

2If the singularity is at $x=x_0$ then we use $(x-x_0)$ in place of x in our series solution.

where *r* is an arbitrary constant. We plug this form into the ODE and solve for *r* and the coefficients a_n. In general two values of *r* will be found, say r_1 and r_2. Then we will have obtained two solutions of the form

$$\begin{aligned} y_1(x) &= x^{r_1}\sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= x^{r_2}\sum_{n=0}^{\infty} b_n x^n \end{aligned} \tag{32}$$

Assume the first non-zero coefficients are a_p and b_q. Then for $x\to 0$

$$\begin{aligned} y_1(x) &\sim a_p x^{r_1+p} \\ y_2(x) &\sim b_q x^{r_2+q} \end{aligned} \tag{33}$$

If $y_1(x), y_2(x)$ are linearly dependent then for $x\to 0$

$$\frac{y_1(x)}{y_2(x)} \sim \frac{a_p}{b_q} x^{r_1-r_2+p-q} = c \tag{34}$$

where *c* is a constant. This requires that $r_1-r_2+p-q=0$, so $r_1-r_2=q-p$. Since *p* and *q* are integers, we must have that r_1-r_2 is an integer. Therefore, if r_1-r_2 is not an integer, the two solutions are linearly independent. If r_1-r_2 is an integer, the two solutions may, or may not, be linearly independent. If the two solutions are linearly dependent, then we can find a second solution of the form

$$y_2(x) = \ln(x)\, y_1(x) + u(x) \tag{35}$$

where

$$u(x) = x^{r_2}\sum_{n=0}^{\infty} b_n x^n \tag{36}$$

Convergence of Series

When representing functions by power series such as

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{37}$$

the question naturally arises: Does the series *converge*? Let's define what we mean by this. Consider the sequence of functions

$$y_k(x) = \sum_{n=0}^{k} a_n x^n \tag{38}$$

If given any $\delta > 0$ you can find an *N* such that

$$|y_k(x) - y(x)| < \delta \tag{39}$$

for all $k > N$ then the sequence of functions *converges* to the limit $y(x)$. This may be true for all *x* or it may be true only for certain values of *x*. In practical terms this means that the function can be approximated by a polynomial of order *N* to within an error of δ over some interval of the *x* axis.

Consider any sequence of numbers s_n and the sequence of summations

$$S_k = \sum_{n=0}^{k} s_n \tag{40}$$

If we can always find an *N* such that for any $\delta > 0$ we have $|S_k - S| < \delta$ whenever $k > N$ then we say the sequence of summations converges, and we write

$$S = \sum_{n=0}^{\infty} s_n \tag{41}$$

If $\sum_{n=0}^{\infty} |s_n|$ converges then we say that the series *converges absolutely*. Since

$$\left|\sum_{n=0}^{\infty} s_n\right| \le \sum_{n=0}^{\infty} |s_n| \tag{42}$$

a series which converges absolutely also converges. Note that if $|s_n| \le |t_n|$, where t_n is some other sequence, then

$$\sum_{n=0}^{\infty} |s_n| \le \sum_{n=0}^{\infty} |t_n| \tag{43}$$

Therefore if each term in a series has magnitude less than the magnitude of each term in some other, absolutely convergent series, that first series is also absolutely convergent.

Consider

$$\begin{aligned} (1-x)(1+x+x^2+\cdots+x^k) &= (1+x+x^2+\cdots+x^k) \\ &\quad -(x+x^2+x^3+\cdots+x^{k+1}) \\ &= 1-x^{k+1} \end{aligned} \tag{44}$$

from which

$$\sum_{n=0}^{k} x^n = \frac{1-x^{k+1}}{1-x} \tag{45}$$

If $|x| < 1$ then $\lim_{k\to\infty} x^{k+1} \to 0$ and we obtain the *geometric series*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{46}$$

which is absolutely convergent for $|x| < 1$. Therefore, if for some sequence we have

$$|s_n| \le A q^n \tag{47}$$

were *A* is a positive number and $0 < q < 1$, then we can write

$$\sum_{n=0}^{\infty} |s_n| \le A \sum_{n=0}^{\infty} q^n = \frac{A}{1-q} \tag{48}$$

It follows that the series $\sum_{n=0}^{\infty} s_n$ is absolutely convergent. This is called the *comparison test*. It also follows that

$$\left|\sum_{n=N}^{\infty} s_n\right| \le \sum_{n=N}^{\infty} |s_n| \le A \sum_{n=N}^{\infty} q^n = A\frac{q^N}{1-q} \tag{49}$$

This gives an upper bound for the error that results if we sum only the first *N* terms of the series. Since in numerical practice we can only sum a finite number of terms, this is very useful to know.

Equivalently, the series is absolutely convergent if there exists an N such that for all $n>N$

$$\left|\frac{s_{n+1}}{s_n}\right|\le q<1 \tag{50}$$

This is called the *ratio test* for convergence. As an example, consider the power series

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \tag{51}$$

Applying the ratio test we have

$$\left|\frac{\frac{x^{(n+1)}}{(n+1)!}}{\frac{x^n}{n!}}\right|=\left|\frac{x}{n+1}\right| \tag{52}$$

If we take any $N\ge|x|$ then clearly for $n>N$ this ratio will be less than unity. Therefore the series is absolutely convergent for all x. If we approximate the series by its first N terms

$$e^x\approx\sum_{n=0}^{N-1}\frac{x^n}{n!} \tag{53}$$

we have the error bound

$$\sum_{n=N}^{\infty}\frac{x^n}{n!}\le\frac{x^N}{N!}\sum_{n=0}^{\infty}q^n=\frac{x^N}{N!}\frac{1}{1-q} \tag{54}$$

where

$$q=\left|\frac{x}{N+1}\right| \tag{55}$$

References

1. Kreyszig, *Advanced Engineering Mathematics*, 4th Ed.,Wiley, 1979, ISBN 0-471-020140-7.