Lecture 1b

Differential operators and orthogonal coordinates

Partial derivatives

Recall from your calculus courses that the derivative of a function can be defined as

$$\frac{d}{dx}f(x)=\lim_{\delta\to 0}\frac{f(x+\delta)-f(x)}{\delta} \tag{1}$$

or using the *central difference* form:

$$\frac{d}{dx}f(x)=\lim_{\delta\to 0}\frac{f(x+\delta/2)-f(x-\delta/2)}{\delta} \tag{2}$$

The *partial derivative* of a function of several variables is defined in a similar manner by varying only one of the variables, such as

$$\frac{\partial}{\partial y}f(x,y)=\lim_{\delta\to 0}\frac{f(x,y+\delta)-f(x,y)}{\delta} \tag{3}$$

The other variables remain fixed. Multiple derivatives can be defined accordingly. For example,

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x,y)=\lim_{\delta\to 0}\frac{\frac{\partial f}{\partial y}(x+\delta,y)-\frac{\partial f}{\partial y}(x,y)}{\delta} \tag{4}$$

where $\partial f/\partial y$ is defined in (3). From the definition it follows that the order in which the derivatives are taken does not matter[1]. For example

$$\frac{\partial^2}{\partial x\partial y}f=\frac{\partial^2}{\partial y\partial x}f \tag{5}$$

because in both cases we have the limit as $\delta\to 0$ of

$$\frac{f(x+\delta,y+\delta)+f(x,y)-f(x+\delta,y)-f(x,y+\delta)}{\delta^2} \tag{6}$$

Gradient

Assume a scalar field $f(x,y,z)$ is defined in some region of space. If we start at the point (x,y,z) and move a displacement $\mathbf{dl}=\hat{a}_x dx+\hat{a}_y dy+\hat{a}_z dz$ away, the function value will change from f to $f+df$. The change is given by

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz \tag{7}$$

This motivates us to define a vector field, which we call the *gradient* of *f*, by the formula

$$\nabla f=\hat{a}_x\frac{\partial f}{\partial x}+\hat{a}_y\frac{\partial f}{\partial y}+\hat{a}_z\frac{\partial f}{\partial z} \tag{8}$$

We can abstractly think of the *del operator* as

$$\nabla=\hat{a}_x\frac{\partial}{\partial x}+\hat{a}_y\frac{\partial}{\partial y}+\hat{a}_z\frac{\partial}{\partial z} \tag{9}$$

The del operator applied to a scalar field produces the gradient.

For any displacement **dl** we can write

$$df=\nabla f\cdot\mathbf{dl} \tag{10}$$

For a fixed length displacement (*dl*) this dot product will be greatest if **dl** is parallel to the gradient. Therefore the direction of the gradient is the direction of the maximum rate of change of the scalar field with respect to displacement, and the magnitude of the gradient is the corresponding rate of change. This gives us a physical interpretation of the gradient that is independent of the coordinate system.

Divergence and divergence theorem

The "dot product" of the del operator with a vector field gives a scalar field called the *divergence* of the vector field. We have

$$\nabla\cdot\mathbf{A}=\frac{\partial}{\partial x}A_x+\frac{\partial}{\partial y}A_y+\frac{\partial}{\partial z}A_z \tag{11}$$

We denote the *flux* of *A* through the closed surface *S* by

$$\psi_A=\oint\!\!\oint_S\mathbf{A}\cdot\mathbf{ds} \tag{12}$$

The *divergence theorem* states that the surface flux is the volume integral of the divergence

$$\boxed{\iiint_V\nabla\cdot\mathbf{A}\,dv=\oint\!\!\oint_S\mathbf{A}\cdot\mathbf{ds}} \tag{13}$$

If our volume is small enough that $\nabla\cdot\mathbf{A}$ is effectively constant, then the volume integral reduces to $V\nabla\cdot\mathbf{A}$ and

$$\nabla\cdot\mathbf{A}\approx\frac{1}{V}\oint\!\!\oint_S\mathbf{A}\cdot\mathbf{ds} \tag{14}$$

We have the physical interpretation that divergence is surface flux per unit volume. Note that this interpretation is independent of any particular coordinate system. The mathematical expression, however, will depend on the

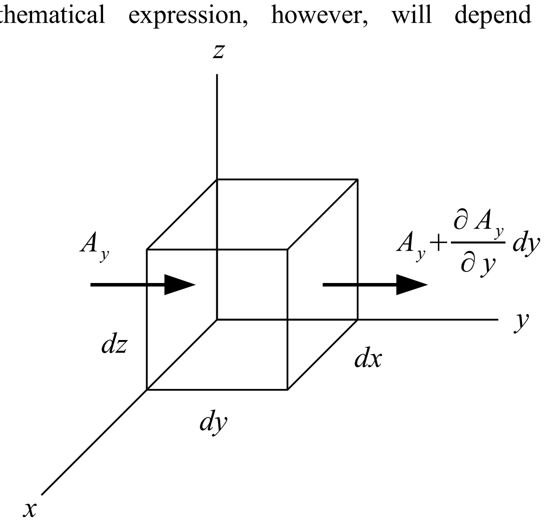

Figure 1: Divergence for a small cubic surface.

1 Provided f satisfies some basic continuity/differentiability conditions.

coordinate system.

Let's consider the divergence theorem in rectangular coordinates applied to a box of dimensions *dx, dy, dz*. If A_y is the component of **A** "entering" the *x,z* face at $y=0$, then we have an inward flux of $A_y\,dx\,dz$. At $y=dy$ the field is $A_y+(\partial A_y/\partial y)dy$ so we have an outward flux of $[A_y+(\partial A_y/\partial y)dy]dx\,dz$. The net outward flux is $(\partial A_y/\partial y)dx\,dy\,dz=(\partial A_y/\partial y)dv$. Repeating in the *x* and *z* directions we obtain for the flux.

$$\nabla\cdot\mathbf{A}\,dv=\left(\frac{\partial}{\partial x}A_x+\frac{\partial}{\partial y}A_y+\frac{\partial}{\partial z}A_z\right)dx\,dy\,dz \tag{15}$$

Curl and Stoke's theorem

The cross product of the del operator with a vector field gives a vector called the *curl* of the vector field.

$$\begin{aligned}\nabla\times\mathbf{A}= \;&\hat{a}_x\left(\frac{\partial}{\partial y}A_z-\frac{\partial}{\partial z}A_y\right)\\ &+\hat{a}_y\left(\frac{\partial}{\partial z}A_x-\frac{\partial}{\partial x}A_z\right)\\ &+\hat{a}_z\left(\frac{\partial}{\partial x}A_y-\frac{\partial}{\partial y}A_x\right)\end{aligned} \tag{16}$$

Stoke's theorem states that the integral of a vector field around a closed loop equals the integral of the curl over any surface bounded by that loop, or

$$\boxed{\iint_S(\nabla\times\mathbf{A})\cdot\mathbf{ds}=\oint_L\mathbf{A}\cdot\mathbf{dl}} \tag{17}$$

For a very small, flat surface of area S over which $\nabla\times\mathbf{A}$ can be considered constant, we have $\iint_S(\nabla\times\mathbf{A})\cdot\mathbf{ds}\approx(\nabla\times\mathbf{A})\cdot\mathbf{S}$. This is illustrated below. If we call $\oint_L\mathbf{A}\cdot\mathbf{dl}$ the *rotation* of **A** about the curve *L* then the physical significance of the curl is that $|\nabla\times\mathbf{A}|$ is the maximum rotation per unit area, and the rotation has this value in the plane normal to $\nabla\times\mathbf{A}$.

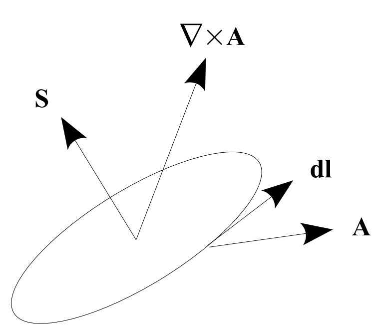

Figure 2: Curl of a vector field.

Notice that if $\mathbf{A}=\hat{a}_zA_z$ everywhere then $\hat{a}_z\cdot\nabla\times\mathbf{A}=0$. In general, if the direction of **A** is constant then $\nabla\times\mathbf{A}$ has no component parallel to **A** and $\mathbf{A}\cdot\nabla\times\mathbf{A}=0$.

Keep in mind that $\mathbf{A}\cdot\nabla\times\mathbf{A}=0$ is not an identity. For example, at a single point in space we might have $A_x=A_y=0$ giving $\mathbf{A}=\hat{a}_zA_z$, but if $\left(\frac{\partial}{\partial x}A_y-\frac{\partial}{\partial y}A_x\right)\neq0$ then $\nabla\times\mathbf{A}$ *will* have a *z* component. The condition $\mathbf{A}\cdot\nabla\times\mathbf{A}=0$ holds everywhere only if the direction of **A** is the same everywhere.

In the above illustration, we will get the most rotation about the curve if we orient **S** to be parallel to $\nabla\times\mathbf{A}$. For this orientation we have $|\nabla\times\mathbf{A}|\approx\frac{1}{S}\oint_L\mathbf{A}\cdot\mathbf{dl}$. The magnitude of the curl is the maximum rotation per unit area, and the rotation is maximum in a plane normal to the direction of $\nabla\times\mathbf{A}$. This physical interpretation is independent of any particular coordinate system. As always, the mathematical expression of this interpretation will depend on the coordinate system employed.

A possible misconception of the curl of a vector field is that $\nabla\times\mathbf{A}\neq0$ implies that the direction of the field **A** somehow "curls around" the vector $\nabla\times\mathbf{A}$. This is not necessarily the case. For example, $\mathbf{A}=\hat{a}_yx^2$ gives $\nabla\times\mathbf{A}=\hat{a}_z(2x)$, but the direction of **A** does not make circles around the *z* axis. Instead, if we integrate $\mathbf{A}\cdot\mathbf{dl}$ around a circle we get a non-zero value.

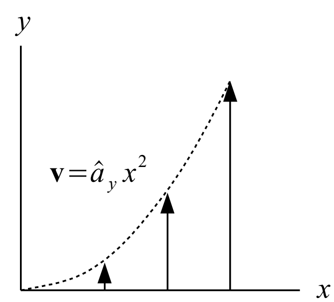

Figure 3: This vector field **v** *has curl* $\hat{a}_z(2x)$.

The derivation of the *z* component of the curl is shown below.

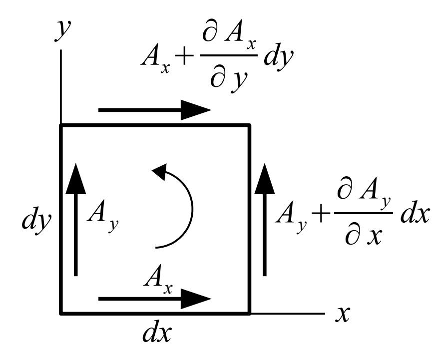

Figure 4: $\oint\mathbf{A}\cdot\mathbf{dl}$ *for a small rectangle in the x-y plane.*

If we integrate $\mathbf{A}\cdot\mathbf{dl}$ counter-clockwise around the rectangle of dimensions *dx, dy* we get a positive contribution A_xdx on the bottom size and a negative contribution $-(A_x+\partial A_x/\partial y\,dy)dx$ on the top side. The net is $-\partial A_x/\partial y\,dy\,dx$. Likewise, we get a positive contribution

$(A_y + \partial A_y / \partial x\, dx) dy$ from the right side and a negative contribution $-A_y dy$. The net is $\partial A_y / \partial x\, dx\, dy$. The total line integral is $(\partial A_y / \partial x - \partial A_x / \partial y) dx\, dy$. This is $(\nabla \times \mathbf{A}) \cdot \hat{a}_z dx\, dy$.

Laplacian

The divergence of the gradient of a scalar function is called the *Laplacian* of the function and is denoted by $\nabla^2 f$. We have

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{18}$$

In rectangular coordinates the Laplacian is the sum of second derivatives. The second derivative is the derivative of the derivative, and using the central-difference definition of a derivative we have (valid as $\delta \to 0$)

$$\frac{d^2}{dx^2} f(x) = \frac{d}{dx} \frac{d}{dx} f(x) \approx \frac{\frac{df}{dx}(x + \delta/2) - \frac{df}{dx}(x - \delta/2)}{\delta} \tag{19}$$

Using the central-difference definition again gives us

$$\begin{aligned} \frac{d^2}{dx^2} f(x) &\approx \frac{\left(\frac{f(x+\delta) - f(x)}{\delta}\right) - \left(\frac{f(x) - f(x-\delta)}{\delta}\right)}{\delta} \\ &= \frac{2}{\delta^2} \left\{ \frac{[f(x+\delta) + f(x-\delta)]}{2} - f(x) \right\} \end{aligned} \tag{20}$$

We see that the second derivative is related to the difference between the function value at a point and the average value nearby.

Doing the same for the *y* and *z* coordinates we obtain

$$\nabla^2 f \approx \frac{6}{\delta^2} \times \left[\frac{1}{6} \begin{pmatrix} f(x+\delta, y, z) + f(x-\delta, y, z) \\ + f(x, y+\delta, z) + f(x, y-\delta, z) \\ + f(x, y, z+\delta) + f(x, y, z-\delta) \end{pmatrix} - f(x, y, z) \right] \tag{21}$$

Therefore, the Laplacian of a function measures the difference between the function at a point and the function's average value at neighboring points. More precisely, the Laplacian is $6/\delta^2$ times the difference between the average value of the function on a sphere of (very small) radius δ and the value of the function at the sphere center. Indeed, if $\nabla^2 f = 0$ everywhere, then the average value of *f* over a sphere (of any size) is equal to the value of *f* at the sphere's center (this is called "Gauss's harmonic function theorem").

In *rectangular coordinates*, the Laplacian of a vector is defined as a vector whose components are the Laplacians of the corresponding components of the vector.

$$\nabla^2 \mathbf{A} \equiv \hat{a}_x \nabla^2 A_x + \hat{a}_y \nabla^2 A_y + \hat{a}_z \nabla^2 A_z \tag{22}$$

Note that this definition is specific to rectangular coordinates. In other coordinate systems the Laplacian of a vector is not so simple. We will need to revisit this when we consider spherical coordinates.

General orthogonal coordinates

Consider any *orthogonal* coordinate system where spatial position **r** is determined by the three coordinates u, v, w. We typically specify three functions $x(u, v, w)$, $y(u, v, w)$ and $z(u, v, w)$ that give the rectangular coordinates as functions of u, v, w. As an example

$$\begin{aligned} x &= u \sin(v) \cos(w) \\ y &= u \sin(v) \sin(w) \\ z &= u \cos(v) \end{aligned} \tag{23}$$

defines the spherical coordinates u, v, w (which we usually denote as r, θ, ϕ).

By orthogonal we mean that the unit vectors

$$\hat{a}_u = \frac{\partial \mathbf{r} / \partial u}{|\partial \mathbf{r} / \partial u|} \quad \hat{a}_v = \frac{\partial \mathbf{r} / \partial v}{|\partial \mathbf{r} / \partial v|} \quad \hat{a}_w = \frac{\partial \mathbf{r} / \partial w}{|\partial \mathbf{r} / \partial w|} \tag{24}$$

are orthogonal at all points in space, that is, $\hat{a}_u \cdot \hat{a}_v = \hat{a}_u \cdot \hat{a}_w = \hat{a}_v \cdot \hat{a}_w = 0$ everywhere.

Let's define

$$h_u = \left| \frac{\partial \mathbf{r}}{\partial u} \right| \tag{25}$$

and likewise for h_v, h_w. We call these the *metric coefficients*. The physical significance of h_u is that the length of the displacement $d\mathbf{r}$ due to a change in the coordinate u of magnitude du is $dl = h_u du$.

Let the coordinates change by the differential amounts du, dv, dw. The length of the resulting displacement, call it *dl*, will be given by the Pythagorean formula[2]

$$dl^2 = h_u^2 du^2 + h_v^2 dv^2 + h_w^2 dw^2 \tag{26}$$

For example, in rectangular coordinates

$$dl^2 = dx^2 + dy^2 + dz^2 \tag{27}$$

and we see that $h_x = h_y = h_z = 1$. In *cylindrical coordinates*

$$dl^2 = (1)^2 d\rho^2 + (\rho)^2 d\phi^2 + (1)^2 dz^2 \tag{28}$$

so $h_\rho = h_z = 1, h_\phi = \rho$. In *spherical coordinates*

$$dl^2 = (1)^2 dr^2 + (r)^2 d\theta^2 + (r \sin\theta)^2 d\phi^2 \tag{29}$$

so $h_r = 1, h_\theta = r, h_\phi = r \sin\theta$. The metric coefficients determine the specific forms that the differential operators take, as we will now see.

Gradient

The gradient is a "directional derivative." The value of $(\nabla f) \cdot \hat{a}$ is equal to df/dl in the direction of $\hat{a}$. In the

2 The Pythagorean theorem applies only if the coordinates are orthogonal. Non-orthogonal coordinates would have cross-terms such as $h_{uv} du\, dv$.

direction $\hat{a}_u$, $dl = h_u du$, so the u component of ∇f is $(\partial f/\partial u)/h_u$ and similarly for the v and w components. Therefore

$$\boxed{\nabla f = \hat{a}_u \frac{1}{h_u}\frac{\partial f}{\partial u} + \hat{a}_v \frac{1}{h_v}\frac{\partial f}{\partial v} + \hat{a}_w \frac{1}{h_w}\frac{\partial f}{\partial w}} \qquad (30)$$

This expresses the gradient in any orthogonal coordinate system.

Divergence

The physical significance of the divergence is that $\nabla\cdot\mathbf{A}$ is the net outward flux per unit volume of the vector field **A** from an infinitesimal volume *dV*. Consider a volume produced by the coordinate changes du, dv, dw. The lengths of the sides will be $h_u du, h_v dv, h_w dw$ so the volume will be $dV = h_u h_v h_w du\,dv\,dw$ (note the distinction between *dV* and *dv*). Consider the flux in the *u* direction. The flux into the volume will be $h_v dv\, h_w dw\, A_u$ evaluated at *u*. The flux out of the volume will be $h_v dv\, h_w dw\, A_u$ evaluated at $u+du$. The next flux in the $\hat{a}_u$ direction is

$$\begin{aligned}\left[(h_v h_w A_u)_{u+du} - (h_v h_w A_u)_u\right] dv\,dw \\ = \frac{\partial}{\partial u}(h_v h_w A_u)\, du\,dv\,dw\end{aligned} \qquad (31)$$

We can write this as

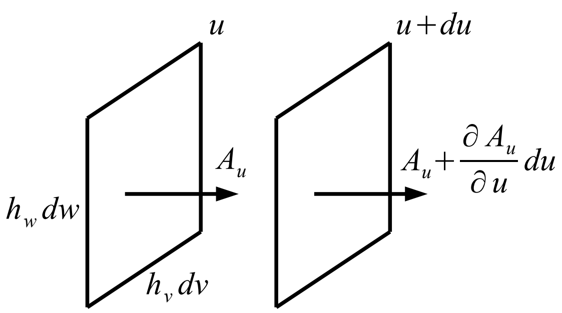

Figure 5: Calculating the u contribution to divergence.

$$\frac{1}{h_u h_v h_w}\frac{\partial}{\partial u}(h_v h_w A_u)\, dV \qquad (32)$$

Applying the same idea to the remaining coordinates we have

$$\boxed{\begin{aligned}\nabla\cdot\mathbf{A} = \;& \frac{1}{h_u h_v h_w}\frac{\partial}{\partial u}(h_v h_w A_u) \\ &+ \frac{1}{h_u h_v h_w}\frac{\partial}{\partial v}(h_w h_u A_v) \\ &+ \frac{1}{h_u h_v h_w}\frac{\partial}{\partial w}(h_u h_v A_w)\end{aligned}} \qquad (33)$$

as our expression for the divergence in general orthogonal coordinates.

Laplacian

Since the Laplacian is the divergence of the gradient, we can determine its expression using our two previous results. Substituting the *u,v,w* components of (30) for A_u, A_v, A_w in (33) we obtain

$$\boxed{\begin{aligned}\nabla^2 f = \;& \frac{1}{h_u h_v h_w}\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) \\ &+ \frac{1}{h_u h_v h_w}\frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v}\frac{\partial f}{\partial v}\right) \\ &+ \frac{1}{h_u h_v h_w}\frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\end{aligned}} \qquad (34)$$

Curl

The curl is the rotation per unit area. Let's consider the component of rotation about the *w* axis. As illustrated below, consider the contribution of A_v to this.

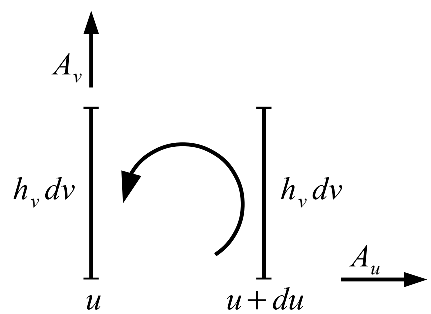

Figure 6: Calculating the A_v contribution to the w component of curl.

For rotation in the direction shown, at $u+du$ we have a contribution to $\oint \mathbf{A}\cdot\mathbf{dl}$ of $(A_v h_v dv)_{u+du}$. At u the contribution is $-(A_v h_v dv)_u$. The total contribution due to A_v is $(A_v h_v dv)_{u+du} - (A_v h_v dv)_u = \frac{\partial}{\partial u}(h_v A_v)\,du\,dv$. A_u contributes $(A_u h_u du)_v - (A_u h_u du)_{v+dv} = -\frac{\partial}{\partial v}(h_u A_u)\,du\,dv$ The area is $h_u h_v du\,dv$ so the *w* component of $\nabla\times\mathbf{A}$ is

$$\frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}(h_v A_v) - \frac{\partial}{\partial v}(h_u A_u)\right] \qquad (35)$$

Doing the same for the *u* and *v* components we obtain

$$\boxed{\begin{aligned}\nabla\times\mathbf{A} = \;& \frac{\hat{a}_u}{h_v h_w}\left[\frac{\partial}{\partial v}(h_w A_w) - \frac{\partial}{\partial w}(h_v A_v)\right] \\ &+ \frac{\hat{a}_v}{h_w h_u}\left[\frac{\partial}{\partial w}(h_u A_u) - \frac{\partial}{\partial u}(h_w A_w)\right] \\ &+ \frac{\hat{a}_w}{h_u h_v}\left[\frac{\partial}{\partial u}(h_v A_v) - \frac{\partial}{\partial v}(h_u A_u)\right]\end{aligned}} \qquad (36)$$

Useful differential operator identities

The following identities can be directly verified. The divergence of the curl of a vector field is identically zero

$$\nabla\cdot(\nabla\times\mathbf{A})=0 \tag{37}$$

The curl of the gradient of a scalar field is identically zero.

$$\nabla\times\nabla f=0 \tag{38}$$

Recall the product rule for scalar derivatives: $(fg)'=fg'+f'g$. Here are product rules involving gradient, divergence and curl.

$$\nabla(fg)=f\nabla g+(\nabla f)g \tag{39}$$

$$\nabla\cdot(f\boldsymbol{A})=f\nabla\cdot\boldsymbol{A}+\nabla f\cdot\boldsymbol{A} \tag{40}$$

$$\nabla\times(f\boldsymbol{A})=f\nabla\times\boldsymbol{A}+\nabla f\times\boldsymbol{A} \tag{41}$$

The following formula for the divergence of a cross product will be important for us

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\mathbf{B} \tag{42}$$

We will make extensive use of the following identity

$$\nabla\times\nabla\times\mathbf{A}=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A} \tag{43}$$

This can be directly verified in rectangular coordinates. We can rewrite this as

$$\nabla^2\mathbf{A}\equiv\nabla(\nabla\cdot\mathbf{A})-\nabla\times\nabla\times\mathbf{A} \tag{44}$$

This will serve as the definition of the Laplacian of a vector field in any coordinate system. In rectangular coordinates it works out to be (22), but for a general system of orthogonal coordinates u,v,w it does not work out so "cleanly." In general

$$\nabla^2\mathbf{A}\neq\hat{a}_u\nabla^2A_u+\hat{a}_v\nabla^2A_v+\hat{a}_w\nabla^2A_w \tag{45}$$

In non-rectangular systems we must use (44) to describe the Laplacian of a vector.

References

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5. mathworld.wolfram.com/OrthogonalCoordinateSystem.html

Appendix – Orthogonal coordinate systems

The Helmholtz equation is known to be separable in eight coordinate systems in addition to rectangular, cylindrical and spherical coordinates [4]. Some of these are

Elliptic cylindrical coordinates

$$\begin{aligned}x&=a\cosh u\cos v\\ y&=a\sinh u\sin v\\ z&=w\end{aligned}$$

Here a is a fixed parameter chosen to give a desired elliptical geometry and u,v,w are the coordinates. These are similar to (circular) cylindrical coordinates except that radial distance ρ has been replaced by $a\cosh u$ for the x coordinate and $a\sinh u$ for the y coordinate. For fixed u,

$$\left(\frac{x}{a\cosh u}\right)^2+\left(\frac{y}{a\sinh u}\right)^2=\cos^2 v+\sin^2 v=1$$

is the equation of an ellipse with semi-major/minor axes $a\cosh u$ and $a\sinh u$. For fixed v,

$$\left(\frac{x}{a\cos v}\right)^2-\left(\frac{y}{a\sin v}\right)^2=\cosh^2 u-\sinh^2 u=1$$

is the equation of a hyperbola.

Bipolar cylindrical coordinates

$$\begin{aligned}x&=\frac{a\sinh u}{\cosh u-\cos v}\\ y&=\frac{a\sin v}{\cosh u-\cos v}\\ z&=w\end{aligned}$$

Again, a is a fixed parameter. A surface of constant u is a circular cylinder of radius $r=|a/\sinh u|$ and center $x=a\cosh u/\sinh u,\ y=0$. These coordinates are useful for describing a two-wire transmission line.

Spheroidal coordinates

There are two spheroidal coordinate system related to spherical coordinates by "stretching" or "compressing" the z coordinate. The *prolate spheroidal coordinates* are defined by

$$\begin{aligned}x&=a\sinh u\sin v\cos w\\ y&=a\sinh u\sin v\sin w\\ z&=a\cosh u\cos v\end{aligned}$$

these are similar to the spherical coordinates (with $v=\theta,\ w=\phi$) but with $r=a\sinh u$ for the x, y coordinates and $r=a\cosh u$ for the z coordinates. Since $\cosh u>\sinh u$ the z coordinate will be larger (relative to the x,y coordinates) than it would be in spherical coordinates. The result is that a surface of constant u is a spheroid that is stretched in the z direction.

On the other hand, for the *oblate spheroidal coordinates*

$$\begin{aligned}x&=a\cosh u\sin v\cos w\\ y&=a\cosh u\sin v\sin w\\ z&=a\sinh u\cos v\end{aligned}$$

the situation is reversed; the z coordinate is relativity smaller.