# Lecture 1a

## *Complex numbers, phasors and vectors*

## Introduction

This course will require you to apply several concepts you learned in your undergraduate math courses. In some cases, such as complex numbers and phasors, you have probably used these concepts regularly. In other cases, such as vector calculus and series solutions of differential equations, this may be the first time you have had to apply them. Accordingly our first few lectures will be devoted to a mathematical review. As this is a review, we will not derive most of the results and theorems, but merely state them. If you wish to review the derivations, please refer to your undergraduate math/physics texts.

## Complex numbers

The *imaginary unit* $j$ has the property[1]

$$j^2=-1 \tag{1}$$

A general *complex number* can be expressed as

$$z=x+jy \tag{2}$$

where *x* and *y* are real numbers. We call them the *real* and *imaginary* parts of *z* and we write

$$\begin{aligned} x&=\operatorname{Re}\{z\}\\ y&=\operatorname{Im}\{z\} \end{aligned} \tag{3}$$

We can also express a complex number in a *polar format* by writing

$$\begin{aligned} x&=\rho\cos\phi\\ y&=\rho\sin\phi \end{aligned} \tag{4}$$

where $\rho$ and $\phi$, the *modulus* and *argument* (or the *magnitude* and *phase*), are real numbers related to *x* and *y* by[2]

$$\begin{aligned} \rho&=\sqrt{x^2+y^2}\\ \phi&=\tan^{-1}(y/x) \end{aligned} \tag{5}$$

Using (4) we can express a complex number in the form

$$\begin{aligned} z&=\rho(\cos\phi+j\sin\phi)\\ &=\rho e^{j\phi} \end{aligned} \tag{6}$$

where in the last step we've used *Euler's formula*

$$e^{j\phi}=\cos\phi+j\sin\phi \tag{7}$$

Some important values are

$$\begin{aligned} e^{\pm j0}&=1\\ e^{\pm j\pi/2}&=\pm j\\ e^{\pm j\pi}&=-1 \end{aligned} \tag{8}$$

From Euler's formula it follows that

$$\begin{aligned} \cos\phi&=\frac{e^{j\phi}+e^{-j\phi}}{2}\\ \sin\phi&=\frac{e^{j\phi}-e^{-j\phi}}{2j} \end{aligned} \tag{9}$$

We sometimes use the following notation to signify the magnitude and phase of a complex number

$$\begin{aligned} \rho&=|z|\\ \phi&=\angle z \end{aligned} \tag{10}$$

We can then write

$$z=|z|e^{j\angle z} \tag{11}$$

As a shorthand we can leave out the $e^j$. For example, we might write $z=1.5\angle 32^\circ$ (some calculators use this type of notation). This means $z=1.5e^{j32^\circ}=1.5[\cos(32^\circ)+j\sin(32^\circ)]$.

The *conjugate* of a complex number is denoted by $z^*$. It is obtained by changing the sign of the imaginary part

$$z^*=x-jy \tag{12}$$

Since

$$\rho e^{-j\phi}=x-jy \tag{13}$$

we can also obtain the conjugate by changing the sign of the argument. In general the change $j\leftarrow -j$ in an expression produces the conjugate.

Let $z=x+jy$ and $w=u+jv$ be two complex numbers. Their sum and difference are

$$z\pm w=(x\pm u)+j(y\pm v) \tag{14}$$

The real and imaginary parts of the sum/difference of two complex numbers are the sum/difference of the real and imaginary parts of the complex numbers. The product of two complex numbers is

$$\begin{aligned} zw&=(x+jy)(u+jv)\\ &=(xu-yv)+j(xv+yu) \end{aligned} \tag{15}$$

In polar notation we have

$$\begin{aligned} zw&=|z|e^{j\angle z}|w|e^{j\angle w}\\ &=|z||w|e^{j(\angle z+\angle w)} \end{aligned} \tag{16}$$

The magnitude of the product is the product of the magnitudes and the phase of the product is the sum of the phases. Note that

$$zz^*=|z|^2=x^2+y^2 \tag{17}$$

For division we have

1 Electrical engineers use *j* for the imaginary unit instead of *i* as is common in math and physics texts.

2 The inverse tangent must be a “four-quadrant” version such as the atan2(y,x) function in Matlab.

$$\frac{z}{w} = \frac{z}{w}\frac{w^*}{w^*} \\ = \frac{xu+yv}{u^2+v^2} + j\frac{yu-xv}{u^2+v^2} \tag{18}$$

in rectangular form or

$$\frac{z}{w} = \frac{|z|}{|w|} e^{j(\angle z - \angle w)} \tag{19}$$

in polar form. The magnitude of the quotient is the quotient of the magnitudes and the phase of the quotient is the difference of the phases.

The exponential of a complex number is

$$e^z = e^x e^{jy} \\ = e^x(\cos y + j \sin y) \tag{20}$$

When taking the product of exponentials we add the (complex) exponents

$$e^z e^w = e^{z+w} \tag{21}$$

## Fourier Transforms and Phasors

The *Fourier transform* of a function of time $f(t)$ is

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \tag{22}$$

We often call $F(\omega)$ the *spectrum* of $f(t)$; $\omega$ is the *radian frequency*. The *inverse Fourier transform* is

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \frac{d\omega}{2\pi} \tag{23}$$

This shows that any function $f(t)$ can be represented as a *superposition* of functions of the form $e^{j\omega t}$ weighted by the spectrum $F(\omega)$.

*Linear, time-invariant* (LTI) systems have the important property that if a sinusoid of a given frequency is applied as the “input” then the “output” is a sinusoid at the same frequency. All that can change is the magnitude and phase of the sinusoid, not the frequency. This property is very useful for analyzing physical systems. Since we will find ourselves using sinusoidal signals almost exclusively, a bookkeeping technique call *phasor notation* will prove very convenient.

A general sinusoidal signal can be written

$$v(t) = a\cos(\omega t + \phi) \tag{24}$$

where $a$ is the *magnitude*, $\phi$ the *phase* and $\omega$ the radian frequency. We can write

$$a\cos(\omega t + \phi) = \mathrm{Re}\left\{a\, e^{j(\omega t + \phi)}\right\} \\ = \mathrm{Re}\left\{\left[a\, e^{j\phi}\right] e^{j\omega t}\right\} \tag{25}$$

We define the *phasor* of $v(t)$ to be the complex number

$$V = a\, e^{j\phi} \tag{26}$$

Then

$$v(t) = \mathrm{Re}\left\{V e^{j\omega t}\right\} \tag{27}$$

In phasor analysis we use the complex constant $V$ as a representation of the *time-domain* signal $v(t)$. We develop equations and solutions in the *phasor domain*. Any time we want the time-domain signal we multiply by $e^{j\omega t}$ and take the real part.

For a non-sinusoidal signal $v(t)$, we can consider a superposition of phasors at different frequencies:

$$v(t) = \int_{-\infty}^{\infty} \mathrm{Re}\left\{V(\omega) e^{j\omega t}\right\} \frac{d\omega}{2\pi} \\ = \mathrm{Re}\left\{\int_{-\infty}^{\infty} V(\omega) e^{j\omega t} \frac{d\omega}{2\pi}\right\} \tag{28}$$

where $V(\omega)$, the phasor amplitude as a function of frequency, is the spectrum of $v(t)$.

## Vectors

We refer to a single number, either real or complex, as a *scalar*. Scalars can represent physical properties that are specified by an amplitude (and possibly phase), such as pressure or voltage. For properties that have both amplitude/phase and direction, such as force, we employ *vectors*. We will use bold-face letters to represent vectors, for example **A**. When writing by hand one typically represents a vector using a bar or arrow above a letter, as in $\bar{A}$ or $\vec{A}$. Another notation is to underline the letter, as in $\underline{A}$, which represents boldface.

In rectangular coordinates we represent a vector by its three components in the *x*, *y* and *z* directions

$$\mathbf{A} = (A_x, A_y, A_z) \tag{29}$$

The *magnitude* of a vector is a scalar and in rectangular coordinates is given by the Pythagorean formula

$$|\mathbf{A}| = \sqrt{|A_x|^2 + |A_y|^2 + |A_z|^2} \tag{30}$$

We will often represent the magnitude of a vector by the corresponding italic letter, as in $A \equiv |\mathbf{A}|$.

Location in space is usually represented by the *position vector*

$$\mathbf{r} = (x, y, z) \tag{31}$$

In this case *r* represents distance from the origin to the point **r**.

Adding or subtracting vectors is accomplished by adding or subtracting their components

$$\mathbf{A} \pm \mathbf{B} = (A_x \pm B_x, A_y \pm B_y, A_z \pm B_z) \tag{32}$$

This is illustrated below. It can be useful to think of a vector difference as $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$, that is, we flip the direction of the second vector and add.

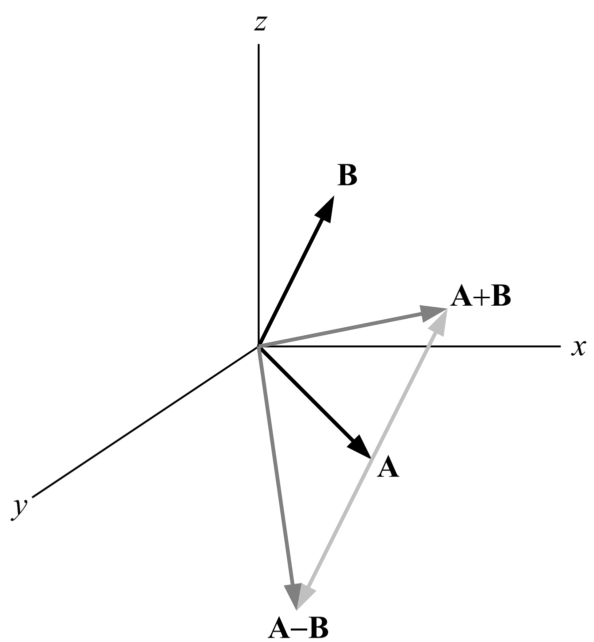


*Figure 1: Vector addition and subtraction.*

*Unit vectors*

If we divide a vector by its magnitude we obtain a vector that has unit magnitude. We will use the following notation for *unit vectors*:

$$\hat{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} \tag{33}$$

The three unit vectors corresponding to the rectangular coordinate axes are

$$\hat{a}_x = (1,0,0) \ ; \ \hat{a}_y = (0,1,0) \ ; \ \hat{a}_z = (0,0,1) \tag{34}$$

These provide an alternate way to represent a vector

$$\mathbf{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z \tag{35}$$

Later we will consider vectors in cylindrical, spherical, and other coordinate systems. In these cases the unit coordinate vectors might be functions of position.

Multiplying a vector by a scalar results in the components of the vector being multiplied by the scalar

$$k\,\mathbf{A} = (kA_x, kA_y, kA_z) \tag{36}$$

*Dot product*

There are two useful ways to combine vectors through multiplication. The *dot product* produces a scalar

$$\mathbf{A}\cdot\mathbf{B} \ = \ A_x B_x + A_y B_y + A_z B_z \tag{37}$$

In terms of magnitudes we have

$$\mathbf{A}\cdot\mathbf{B} \ = \ A\,B\cos\theta_{AB} \tag{38}$$

where $\theta_{AB}$ is the angle between the vectors.

If $\mathbf{A}\cdot\mathbf{B}=0$ for two non-zero vectors, we say they are *orthogonal*. Since $A\neq 0, B\neq 0$, we must have $\cos\theta_{AB}=0$, or $\theta_{AB}=\pi/2=90^{\circ}$.

*Cross product*

The *cross product* of two vectors produces another vector.

$$\begin{aligned}\mathbf{A}\times\mathbf{B} \ = \ & \hat{a}_x(A_y B_z - A_z B_y) + \\ & \hat{a}_y(A_z B_x - A_x B_z) + \\ & \hat{a}_z(A_x B_y - A_y B_x)\end{aligned} \tag{39}$$

Notice the permutations of *xyz* from top to bottom. You can use these to memorize the above formula. The cross product $\mathbf{A}\times\mathbf{B}$ is orthogonal to both **A** and **B**, and the magnitude of the cross product is

$$|\mathbf{A}\times\mathbf{B}| \ = \ A\,B\sin\theta_{AB} \tag{40}$$

This is illustrated below.

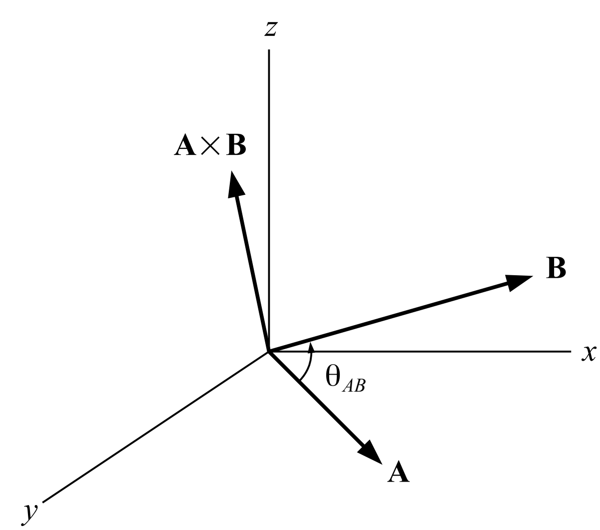


*Figure 2: Cross product of two vectors*

The direction of the cross product is given by the "right hand rule." Point the fingers of your right hand in the direction of **A**. Sweep your fingers to the director of **B**. Your right thumb then points in the direction of $\mathbf{A}\times\mathbf{B}$.

The cross product is zero if $\theta_{AB}=0,\pi$, that is, if the two vectors are *parallel*.

*Normal and tangential components*

We are often interested in the orientation of a vector relative to some surface. Assume the *surface normal* is $\hat{a}_n$. This is a unit vector which is perpendicular to the surface. We can break a vector **A** into normal and tangential components

$$\mathbf{A} = \mathbf{A}_n + \mathbf{A}_t \tag{41}$$

where

$$\mathbf{A}_n = \hat{a}_n(\mathbf{A}\cdot\hat{a}_n)$$
$$\mathbf{A}_t = \mathbf{A} - \mathbf{A}_n = \hat{a}_n \times \mathbf{A} \times \hat{a}_n \tag{42}$$

This is illustrated below.

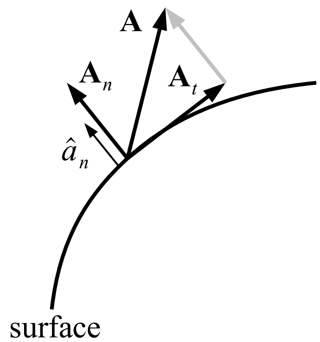


*Figure 3: Components of a vector normal and tangent to a surface.*

Note that the substitution $\hat{a}_n \to -\hat{a}_n$ leaves $\mathbf{A}_n, \mathbf{A}_t$ unchanged. Therefore, it doesn't matter in which direction we define the surface normal.

## Scalar fields

A *scalar field* is a mapping that assigns a scalar to every spatial position in some spatial domain. We denote this with an expression of the form $f(\mathbf{r})$. In rectangular coordinates this would be expressed as $f(x, y, z)$. A scalar field can be complex in which case we represent it as the sum of its real and imaginary parts

$$f(\mathbf{r}) = f_r(\mathbf{r}) + j\, f_i(\mathbf{r}) \tag{43}$$

A scalar field, either real or complex, can also be a function of time. We then write $f(\mathbf{r};t)$. This means that at any given point $\mathbf{r}_0$ the field is a function of time given by $g(t) = f(\mathbf{r}_0;t)$. If at every point in space the field is a real sinusoidal function of time with radian frequency $\omega$ then we can apply the phasor concept. The result is a *scalar phasor field* $f(\mathbf{r})$ where

$$f(\mathbf{r};t) = \mathrm{Re}\{f(\mathbf{r})e^{j\omega t}\} \tag{44}$$

## Vector fields

A *vector field* is a mapping that assigns a vector to every spatial position in some spatial domain, and we write $\mathbf{A}(\mathbf{r})$. Breaking this into rectangular components we have

$$\mathbf{A}(\mathbf{r}) = \hat{a}_x A_x(x, y, z) + \hat{a}_y A_y(x, y, z) + \hat{a}_z A_z(x, y, z) \tag{45}$$

A vector field defines three scalar fields, one for each vector component. The scalar fields, and therefore the vector field, may be complex. We may employ a coordinate system other than rectangular coordinates. We will consider general orthogonal coordinate systems in a subsequent lecture.

We can combine the vector field concept with the phasor concept to arrive at a *vector phasor field*. In this case

$$\mathbf{A}(\mathbf{r};t) = \mathrm{Re}\{\mathbf{A}(\mathbf{r})e^{j\omega t}\} \tag{46}$$

An example is

$$\begin{aligned}\mathbf{A}(\mathbf{r};t) &= \mathrm{Re}\{\hat{a}_x A_0 e^{j\phi} e^{-j\beta z} e^{j\omega t}\} \\ &= \hat{a}_x A_0 \cos(\omega t - \beta z + \phi)\end{aligned} \tag{47}$$

where we assume $A_0$ is real.

We will almost always represent the electric and magnetic fields as vector phasor fields. Indeed, most of this course will involve manipulations of, and relations between, vector phasor fields.

## Time-average dot product

Consider two time-varying, real vectors representing some physical fields

$$(A_x\cos(\omega t + \phi_x),\, A_y\cos(\omega t + \phi_y),\, A_z\cos(\omega t + \phi_z)) \tag{48}$$

and

$$(B_x\cos(\omega t + \theta_x),\, B_y\cos(\omega t + \theta_y),\, B_z\cos(\omega t + \theta_z)) \tag{49}$$

The dot product is

$$\begin{aligned}&A_x B_x \cos(\omega t + \phi_x)\cos(\omega t + \theta_x) + \\ &A_y B_y \cos(\omega t + \phi_y)\cos(\omega t + \theta_y) + \\ &A_z B_z \cos(\omega t + \phi_z)\cos(\omega t + \theta_z)\end{aligned} \tag{50}$$

In most EM problems it is the time-average of such quantities that are usually of interest. Using a basic trig identity

$$\begin{aligned}&A_x B_x \cos(\omega t + \phi_x)\cos(\omega t + \theta_x) \\ &= \frac{1}{2} A_x B_x[\cos(\phi_x - \theta_x) + \cos(2\omega t + \phi_x + \theta_x)]\end{aligned} \tag{51}$$

Time-averaging will eliminate the last term. So the time-average dot product is

$$\frac{A_x B_x}{2}\cos(\phi_x - \theta_x) + \frac{A_y B_y}{2}\cos(\phi_y - \theta_y) + \frac{A_z B_z}{2}\cos(\phi_z - \theta_z) \tag{52}$$

The corresponding phasors are

$$\mathbf{A} = (A_x e^{j\phi_x}, A_y e^{j\phi_y}, A_z e^{j\phi_z}) \tag{53}$$

$$\mathbf{B} = (B_x e^{j\theta_x}, B_y e^{j\theta_y}, B_z e^{j\theta_z}) \tag{54}$$

By inspection we can verify that the time-average dot product we derived above is equal to

$$\frac{1}{2}\mathrm{Re}(\mathbf{A}\cdot\mathbf{B}^*) = \frac{1}{2}\mathrm{Re}(\mathbf{A}^*\cdot\mathbf{B}) \tag{55}$$

Therefore, when expressions like $1/2\,\mathrm{Re}(\mathbf{A}\cdot\mathbf{B}^*)$, where $\mathbf{A}$ and $\mathbf{B}$ are vector phasors, comes up in our analysis, we can identify this as the time-average of the dot product of the corresponding real, physical fields that the phasors represent.

## Line, surface and volume integrals

Two important integration operations that can be performed on a vector field are *line integrals* and *surface integrals*. Suppose we have a vector field **A** and we define a path $L$ between two points $P_{1,}P_2$ , as shown below.

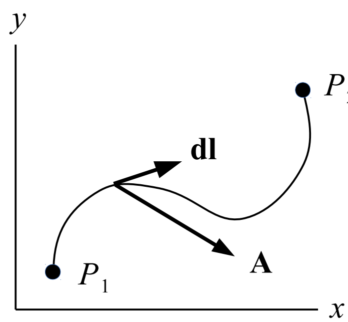


*Figure 4: Line integral*

The line integral of **A** over path $L$ is a scalar denoted by $\int_L \mathbf{A}\cdot\mathbf{dl}$ . The integrand is

$$\mathbf{A}\cdot\mathbf{dl}=A_x dx+A_y dy+A_z dz \tag{56}$$

The displacement **dl** is tangent to the curve. If the path is a *closed path* ( $P_1=P_2$ ) then we use the notation $\oint_L \mathbf{A}\cdot\mathbf{dl}$ .

Now suppose we define some surface $S$, as illustrated below. The surface integral of **A** over $S$ is a scalar denoted by $\iint_S \mathbf{A}\cdot\mathbf{ds}$ . The integrand is

$$\mathbf{A}\cdot\mathbf{ds}=\mathbf{A}\cdot\hat{a}_n\,dS=\mathbf{A}_n\,ds \tag{57}$$

The vector **ds** is normal to the surface and has area $ds$. The surface integral sums the normal component of **A** over the surface. If the surface is a *closed surface*, we use the notation $\oint\!\!\oint_S \mathbf{A}\cdot\mathbf{ds}$ .

Finally, a volume integral of a scalar field is denoted by $\iiint_V f\,dv$ .

## Useful vector identities

The following identities can be readily verified by carrying out the indicated operations in rectangular coordinates.

$$\mathbf{A}\times\mathbf{B}\ =\ -\mathbf{B}\times\mathbf{A} \tag{58}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C} \tag{59}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \tag{60}$$

$$\hat{a}_n\cdot(\mathbf{A}\times\mathbf{B})=\hat{a}_n\cdot(\mathbf{A}_t\times\mathbf{B}_t) \tag{61}$$

This last identity says that the normal component of a cross product is equal to the cross product of the tangential components.

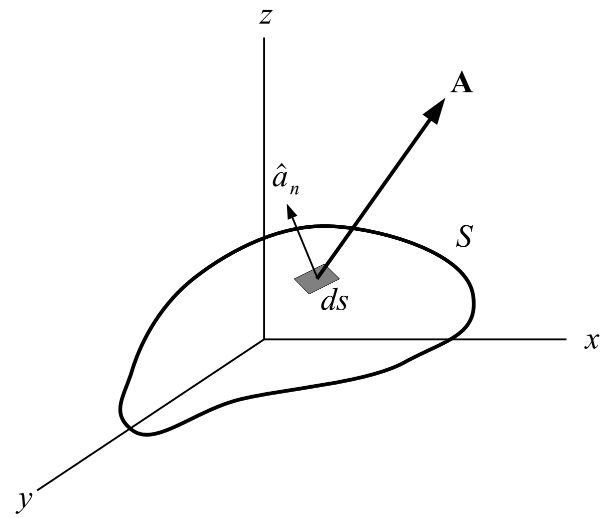


*Figure 5: Surface integral.*

## References


1. Churchill, Brown and Verhey, *Complex Variables and Applications*, 3rd Ed., McGraw Hill, 1976, ISBN 0-07-010855-2.
2. Carrier, Krook and Pearson, *Functions of a complex variable*, Hod Books, 1983, ISBN 07-010089-6.
3. Kreyszig, *Advanced Engineering Mathematics*, 4th Ed.,Wiley, 1979, ISBN 0-471-020140-7.