Lecture 19

Sliding DFT

Definition

Given N samples $x(m)$, $0 \le m \le N-1$, the N-point DFT is defined as

$$X_k = \sum_{m=0}^{N-1} x(m) W_N^{km} \tag{1}$$

with $W_N = e^{-j\frac{2\pi}{N}}$. In this lecture we will be using a subscript instead of parentheses to represent the frequency bin; we will write X_k instead of $X(k)$. We reserve parentheses for a time index.

Now, imagine we have eight samples in $x(n)$, but only want to perform a 4-point DFT. There are five sets of four adjacent samples (Fig. 1) we can perform the DFT on. We use the notation

$$X_k(n) = \sum_{m=0}^{N-1} x(n+m) W_N^{km} \tag{2}$$

to represent the amplitude of the k^{th} frequency bin of the DFT calculated using samples $n \le m \le n+N-1$. We call $X_k(n)$ a *sliding DFT* (SDFT).

Iterative calculation

Consider a five-point DFT. Starting at time $n=0$ we have

$$X_k(0) = x(0) W_5^{k0} + x(1) W_5^{k1} + x(2) W_5^{k2} + x(3) W_5^{k3} + x(4) W_5^{k4} \tag{3}$$

For the next time step $n=1$ we have

$$X_k(1) = x(1) W_5^{k0} + x(2) W_5^{k1} + x(3) W_5^{k2} + x(4) W_5^{k3} + x(5) W_5^{k4} \tag{4}$$

At each time step we have to do four multiplications ($W_5^{k0}=1$ doesn't count) and four additions for each frequency index k. However, look at the $x(2)$ terms in (3) and (4). In the first case the

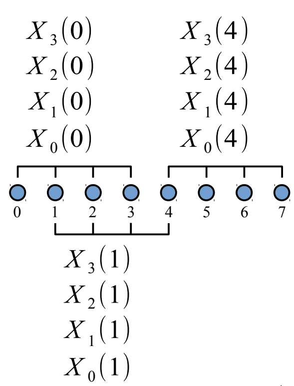

Fig. 1: Sliding DFT. Circles represent samples $x(0), x(1), \ldots, x(7)$. From a given starting sample we perform a 4-point DFT to obtain $X_0, \ldots, X_3$. There are five possible DFTs for this set of samples: $X_k(0), X_k(1), \ldots, X_k(4)$.

coefficient is W_5^{k2} while in the second it is W_5^{k1}. These are related by $W_5^{k1}=W_5^{-k}W_5^{k2}$. Now look at the $x(3)$ terms. These are W_5^{k3} and W_5^{k2} and the relation is $W_5^{k2}=W_5^{-k}W_5^{k3}$. Likewise the $x(4)$ terms have coefficients related by $W_5^{k3}=W_5^{-k}W_5^{k4}$. This structure is apparent if we rewrite the DFTs starting with the common samples

$$\begin{aligned} X_k(0)&=\left[x(1)W_5^{k1}+x(2)W_5^{k2}+x(3)W_5^{k3}+x(4)W_5^{k4}\right]+x(0)W_5^{k0} \\ X_k(1)&=\left[x(1)W_5^{k0}+x(2)W_5^{k1}+x(3)W_5^{k2}+x(4)W_5^{k3}\right]+x(5)W_5^{k4} \end{aligned} \tag{5}$$

Since

$$\begin{aligned} &W_5^{-k}\left[x(1)W_5^{k1}+x(2)W_5^{k2}+x(3)W_5^{k3}+x(4)W_5^{k4}\right] \\ &=\left[x(1)W_5^{k0}+x(2)W_5^{k1}+x(3)W_5^{k2}+x(4)W_5^{k3}\right] \end{aligned} \tag{6}$$

we can write

$$W_5^{-k}X_k(0)=\left[x(1)W_5^{k0}+x(2)W_5^{k1}+x(3)W_5^{k2}+x(4)W_5^{k3}\right]+W_5^{-k}x(0) \tag{7}$$

All the bracketed terms appear in the expression for $X_k(1)$. We need only subtract the last term here and add in the last term of the expression for $X_k(1)$ to get

$$X_k(1)=W_5^{-k}X_k(0)-W_5^{-k}x(0)+x(5)W_5^{k4} \tag{8}$$

Writing

$$W_5^{k4}=W_5^{k(5-1)}=W_5^{-k}W_5^{k5}=W_5^{-k} \tag{9}$$

where we have used $W_5^{k5}=e^{-j\frac{2\pi}{5}k5}=e^{-j2\pi k}=1$ we obtain

$$X_k(1)=W_5^{-k}\left[X_k(0)-x(0)+x(5)\right] \tag{10}$$

Given $X_k(0)$ we need only an addition, a subtraction and one multiplication to obtain $X_k(1)$. This is a significant computational savings.

The general formula for arbitrary N and time index is

$$X_k(n+1)=W_N^{-k}\left[X_k(n)-x(n)+x(n+N)\right] \tag{11}$$

At time step n this expression involves the future sample $x(n+N)$. Typically we prefer to deal with causal systems in which the present does not depend on the future. Subtracting N from all time indices (which is equivalent to shifting the time axis) we obtain

$$X_k(n+1-N)=W_N^{-k}\left[X_k(n-N)-x(n-N)+x(n)\right] \tag{12}$$

Defining

$$y(n)\stackrel{\text{def}}{=}X_k(n+1-N) \tag{13}$$

we have

$$y(n)=e^{j\frac{2\pi}{N}k}\left[y(n-1)+x(n)-x(n-N)\right] \tag{14}$$

The present output depends on the previous output, the current input and the input from N time steps ago. Note that we don't explicitly reference the frequency bin k, but it's implicit in the exponential expression.

Taking z transforms we have

$$Y(z)\left[1-e^{j\frac{2\pi}{N}k}z^{-1}\right]=e^{j\frac{2\pi}{N}k}X(z)\left[1-z^{-N}\right] \tag{15}$$

So, the frequency response is

$$H(\omega)=\frac{1-e^{-jN\omega}}{e^{-j\frac{2\pi}{N}k}-e^{-j\omega}} \tag{16}$$

This expression is undefined at $\omega=2\pi k/N$ where it has a 0/0 form. The limiting value is

$$\lim_{\omega\to\frac{2\pi}{N}k} H(\omega)=N e^{j\frac{2\pi}{N}k} \tag{17}$$

Exercise 1: Prove (17) using L'Hospital's rule.

Let's assume $x(n<0)=0$. We start with

$$y(0)=e^{j\frac{2\pi}{N}k}x(0) \tag{18}$$

Then for $n=1,2,\ldots,N-1$ we set

$$y(n)=e^{j\frac{2\pi}{N}k}\left[y(n-1)+x(n)\right] \tag{19}$$

Finally for $n\geq N$ we have

$$y(n)=e^{j\frac{2\pi}{N}k}\left[y(n-1)+x(n)-x(n-N)\right] \tag{20}$$