

# Lecture 15

## Parametric filters

### Introduction

A *parametric filter* is a filter in which the transfer function coefficients are specified in terms of physically meaningful *parameters* such as center frequency, gain and bandwidth. In this lecture we will limit consideration to the important special case of IIR *biquadratic* filters, commonly called *biquads*. These have transfer functions of the form

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}} \quad (1)$$

Furthermore we will focus on *peaking filters* which *boost* or *cut* the frequency response over some narrow range of frequencies.

### Biquad peaking filters

The frequency response magnitude of a peaking filter (on a dB scale) looks something like that illustrated in Fig. 1.

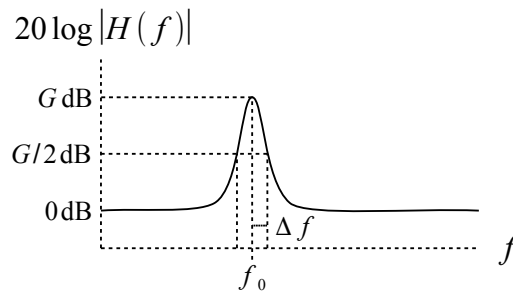


Fig. 1: Magnitude frequency response of peaking filter.

At most frequencies the response is nearly 0 dB, and the filter does not appreciably change these frequency components. At the *center frequency*  $f_0$  the response peaks at  $G$  dB while the response drops to  $G/2$  dB at  $f_0 \pm \Delta f$ . The effect of the filter is to *boost* frequency components within about  $\Delta f$  of the center frequency. If  $G$  is negative then the frequency components are attenuated, or *cut*.

Let's define the linear-scale amplitude corresponding to  $G/2$  dB as

$$A = 10^{(G/2)/20}$$

Then

$$A^2 = 10^{G/20}$$

is the amplitude at  $f_0$ . We will take our filter specifications to be

$$\begin{aligned} |H(f_0)| &= A^2 \\ |H(f_0 \pm \Delta f)| &= A \\ |H(0)| &= |H(0.5)| = 1 \end{aligned} \quad (2)$$

This gives us five constraints on  $H(f)$ . Although  $H(f)$  has six coefficients, one of them is arbitrary since we can always multiply numerator and denominator by an arbitrary constant. Therefore our five constraints should uniquely determine the frequency response of a biquad filter.

Multiplying the numerator and denominator of (1) by  $z$  results in the more symmetric expression

$$H(z) = \frac{b_0 z + b_1 + b_2 z^{-1}}{a_0 z + a_1 + a_2 z^{-1}} \quad (3)$$

Setting  $z = e^{j\omega}$  gives us the frequency response

$$H(\omega) = \frac{b_0 e^{j\omega} + b_1 + b_2 e^{-j\omega}}{a_0 e^{j\omega} + a_1 + a_2 e^{-j\omega}} \quad (4)$$

If we define

$$m_b = \frac{b_0 + b_2}{2}, \delta_b = \frac{b_0 - b_2}{2} \quad (5)$$

then we can write

$$b_0 = m_b + \delta_b, b_2 = m_b - \delta_b \quad (6)$$

and the numerator can be expressed as

$$b_0 e^{j\omega} + b_1 + b_2 e^{-j\omega} = b_1 + 2 m_b \cos \omega + j 2 \delta_b \sin \omega \quad (7)$$

Treating the denominator similarly we have

$$H(\omega) = \frac{b_1 + 2 m_b \cos \omega + j 2 \delta_b \sin \omega}{a_1 + 2 m_a \cos \omega + j 2 \delta_a \sin \omega} \quad (8)$$

where

$$m_a = \frac{a_0 + a_2}{2}, \delta_a = \frac{a_0 - a_2}{2} \quad (9)$$

**Exercise 1:** Verify (6) and (7).

The filter specs at  $f = 0, 0.5$  ( $\omega = 0, \pi$ ) require

$$\frac{b_1 + 2 m_b}{a_1 + 2 m_a} = \frac{b_1 - 2 m_b}{a_1 - 2 m_a} = 1$$

We can satisfy these by setting

$$a_1 = b_1 = c, m_a = m_b = 1 \quad (10)$$

where  $c$  is some constant. The transfer function is now

$$H(\omega) = \frac{c + 2 \cos \omega + j 2 \delta_b \sin \omega}{c + 2 \cos \omega + j 2 \delta_a \sin \omega} \quad (11)$$

At  $\omega = 0, \omega = \pi$  the imaginary parts of the numerator and denominator vanish. If we take  $c = -2 \cos \omega_0$  then the real parts of the numerator and denominator will vanish at  $\omega = \omega_0$ . Using this value of  $c$  and dividing numerator and denominator by 2 results in

$$H(\omega) = \frac{(\cos \omega - \cos \omega_0) + j \delta_b \sin \omega}{(\cos \omega - \cos \omega_0) + j \delta_a \sin \omega} \quad (12)$$

The frequency response at  $\omega = \omega_0$  is

$$H(\omega_0) = \frac{j \delta_b \sin \omega_0}{j \delta_a \sin \omega_0} = \frac{\delta_b}{\delta_a} \quad (13)$$

Our specs require

$$\frac{\delta_b}{\delta_a} = A^2 \quad (14)$$

Finally, we need to satisfy the specs at  $f = f_0 \pm \Delta f$ . Multiplying numerator and denominator by  $-1/\sin \omega$ , and defining

$$u(\omega) = \frac{\cos \omega_0 - \cos \omega}{\sin \omega} \quad (15)$$

we can express the transfer function as

$$H(\omega) = \frac{u - j \delta_b}{u - j \delta_a} \quad (16)$$

The magnitude squared is

$$|H(\omega)|^2 = \frac{u^2 + \delta_b^2}{u^2 + \delta_a^2}$$

Note that  $u(\omega_0) = 0$ . Let  $u = \pm \alpha$  be the values of  $u$  where  $|H(\omega)| = A$  and  $|H(\omega)|^2 = A^2$ . Then

$$\frac{\alpha^2 + \delta_b^2}{\alpha^2 + \delta_a^2} = A^2$$

Clear fractions

$$\alpha^2 + \delta_b^2 = A^2(\alpha^2 + \delta_a^2)$$

Use (14) to write  $\delta_b = A^2 \delta_a$  resulting in

$$\alpha^2 + A^4 \delta_a^2 = A^2(\alpha^2 + \delta_a^2)$$

Moving  $\delta_a$  terms to the left and  $\alpha$  terms to the right

$$(A^4 - A^2) \delta_a^2 = (A^2 - 1) \alpha^2$$

and solving for  $\delta_a^2$  gives us

$$\delta_a^2 = \frac{A^2 - 1}{A^4 - A^2} \alpha^2 = \frac{\alpha^2}{A^2}$$

Finally we have

$$\delta_a = \frac{\alpha}{A}$$

and

$$\delta_b = A^2 \delta_a = \alpha A$$

How does  $u = \pm \alpha$  relate to  $\omega$ ? The first-order Taylor series of (15) is

$$u = (\omega - \omega_0) + \dots \quad (17)$$

Therefore

$$u = \pm \alpha \approx (\omega - \omega_0) \rightarrow \omega \approx \omega_0 \pm \alpha \quad (18)$$

and  $\alpha = \Delta \omega = 2\pi \Delta f$ .

**Exercise 2:** Verify that (15) has Taylor series (17).

In summary: to design a filter with given  $f_0, \Delta f, G_{dB}$  we calculate  $\omega_0 = 2\pi f_0$ ,  $\alpha = 2\pi \Delta f$ ,  $A = 10^{G_{dB}/40}$  and

$$\delta_a = \frac{\alpha}{A}, \delta_b = \alpha A$$

Then the filter coefficients are

$$b_0 = 1 + \delta_b, b_2 = 1 - \delta_b, a_0 = 1 + \delta_a, a_2 = 1 - \delta_a, a_1 = b_1 = -2 \cos \omega_0 \quad (19)$$

An alternate way to specify the bandwidth is using the *Q factor*

$$Q = \frac{\omega_0}{\Delta \omega} = \frac{f_0}{\Delta f} \quad (20)$$

In terms of  $Q$

$$\Delta \omega = \frac{\omega_0}{Q}, \Delta f = \frac{f_0}{Q} \quad (21)$$

For example, with  $Q=10$  the bandwidth is 1/10 of the center frequency. In many applications specifying the fractional bandwidth in this manner is more useful than giving an absolute bandwidth.

A peaking filter with  $G > 0$  dB *boosts* frequencies near  $f_0$  (Fig. 2); a peaking filter with  $G < 0$  dB *cuts* frequencies near  $f_0$  (Fig. 3). In audio applications this can be useful for suppressing “booming” frequency components at the resonance frequencies of a room, speaker cabinet, or musical instrument. An example is shown in Fig. 4. The original music has spectral peaks near 6 kHz and 7.2 kHz due to, respectively, a tambourine and a cymbal. The tambourine is considerably louder than the cymbal which gets lost in the mix. By applying a peaking filter the tambourine volume was reduced by about 4 dB without significantly affecting the rest of the music signal and balanced the tambourine and cymbal in the mix.

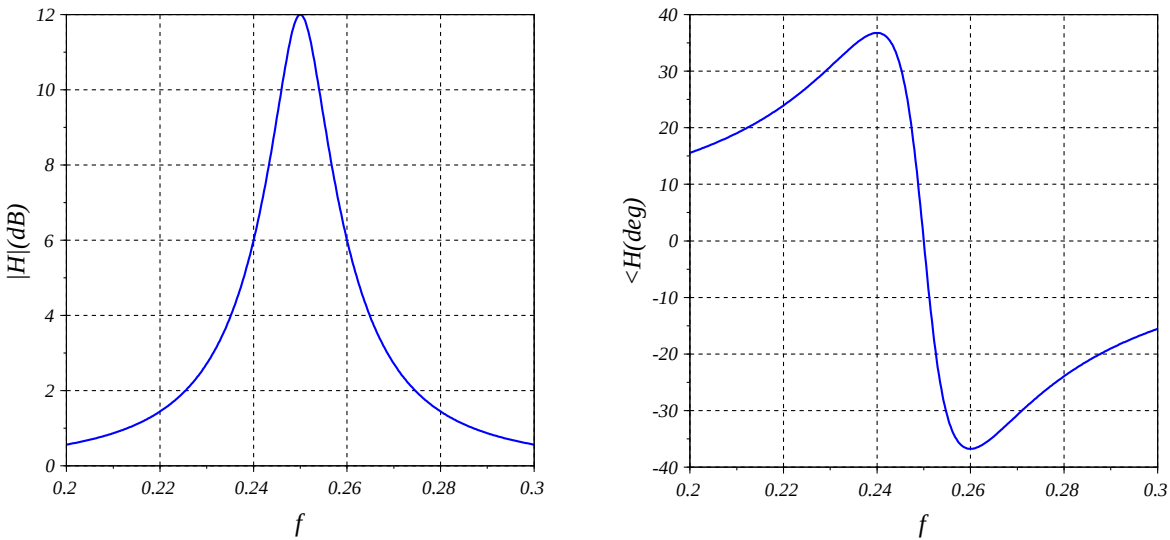


Fig. 2: Frequency response of peaking filter with  $f_0=0.25$ ,  $\alpha=2\pi(0.01)$ ,  $g=12$  dB . Response magnitude is about 6 dB at  $f=f_0\pm 0.01$  . This filter “boosts” frequencies near  $f_0$  . The filter  $Q$  is 25.

## References

1. <http://www.musicdsp.org/files/Audio-EQ-Cookbook.txt>
2. <http://www.earlevel.com/main/2010/12/20/biquad-calculator/>
3. Texas Instruments Application Report SLAA447–April 2010, *Configure the Coefficients for Digital Biquad Filters in TLV320AIC3xxx Family*

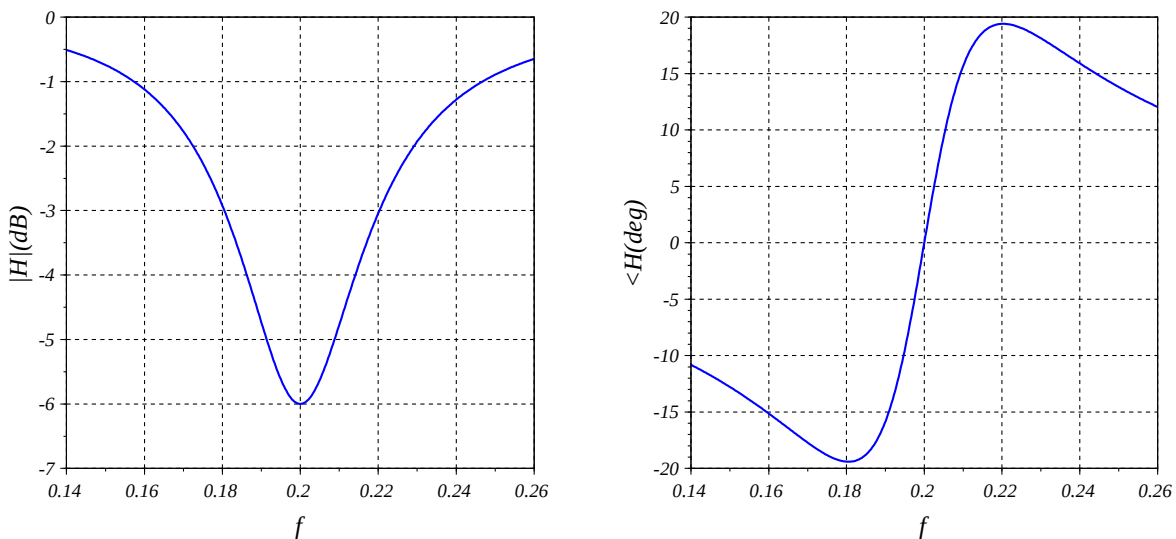


Fig. 3: Frequency response of peaking filter with  $f_0=0.2$ ,  $\alpha=2\pi(0.02)$ ,  $g=-6$  dB . Response magnitude is about  $-3$  dB at  $f=f_0\pm 0.02$  . This filter “cuts” frequencies near  $f_0$  . The filter  $Q$  is 10.

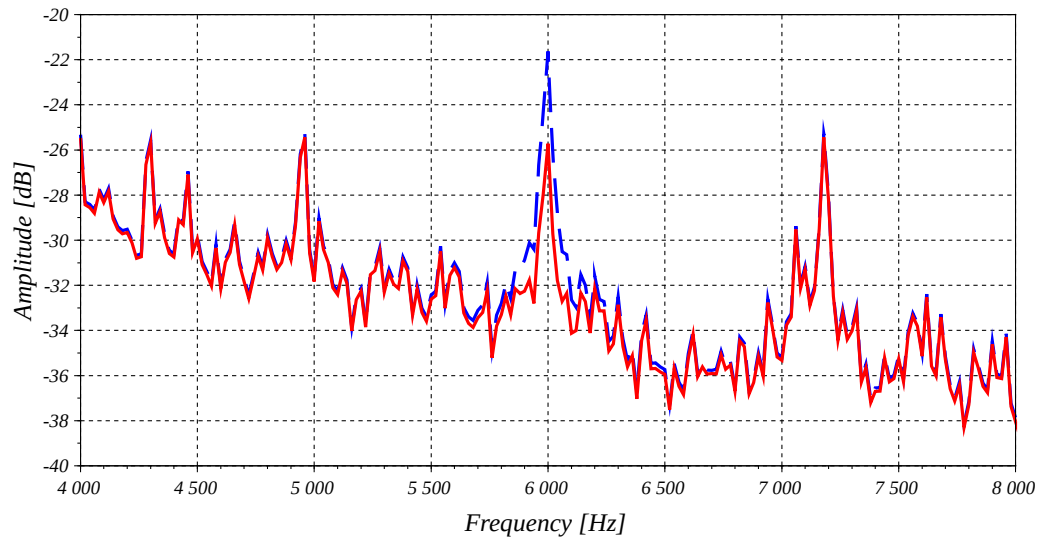


Fig. 4: Peaking filter applied to music to reduce a spectral peak near 6 kHz by 4 dB. Parameters:  $f_0=6\text{kHz}/44.1\text{kHz}$ ,  $\Delta f=60\text{Hz}/44.1\text{kHz}$  ( $Q=100$ ),  $G=-4\text{dB}$ . Dashed line is original spectrum. Solid line is filtered spectrum.

## Appendix – Scilab code

```
function [b,a] = PFquad(f0,df,GdB) //biquad peaking filter
    A = 10^(GdB/20); //linear gain at f0
    da = 2*pi*df/sqrt(A);
    db = da*A;
    c = -2*cos(2*pi*f0);
    b = [1+db,c,1-db];
    a = [1+da,c,1-da];
    b = b/a(1);
    a = a/a(1);
endfunction
```