Lecture 8

Fourier transform

Introduction

The Fourier transform is a special case of the *z* transform. The inverse Fourier transform allows us to represent any signal as a superposition of sinusoids, or "pure tones." Consequently it plays a central role in audio analysis and is somewhat more intuitive than the *z* transform.

Definition

The *z* transform of signal $x(n)$ is

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{1}$$

Evaluating this for $z=e^{j\omega}$ results in the *Fourier transform*

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{2}$$

Although technically incorrect, for convenience we will often use the compact notation

$$X(\omega)\stackrel{\text{def}}{=}X(e^{j\omega}) \ , \ X(f)\stackrel{\text{def}}{=}X(e^{j2\pi f}) \tag{3}$$

when there is no danger of misinterpretation.

Important properties

The Fourier transform is periodic with period 2π because

$$e^{j(\omega\pm 2\pi)}=e^{j\omega} \rightarrow X(\omega\pm 2\pi)=X(\omega) \tag{4}$$

If $x(n)$ is real

$$X(-\omega)=\sum_{n=-\infty}^{\infty} x(n)e^{j\omega n}=X^*(\omega) \tag{5}$$

and the Fourier transform for a negative frequency is just the conjugate of that at the corresponding positive frequency. Therefore all information of $X(f)$ is contained in the frequency interval $0\le f\le 1/2$.

Inverse transform

Because the Fourier transform is simply a special case of the *z* transform, the standard partial-fraction expansion method used for the inverse *z* transform can also be used to evaluate the inverse Fourier transform. In addition, the inverse Fourier transform can be expressed as the integral

$$x(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \tag{6}$$

This form represents the signal $x(n)$ as a superposition of complex exponentials (hence sinusoids) with frequencies in the range $-\pi\le\omega\le\pi$. We can verify (6) directly by substituting (2) to find

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x(k)\int_{-\pi}^{\pi} e^{-j\omega k}e^{j\omega n}d\omega\end{aligned} \quad (7)$$

Since

$$\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega = 2\pi\delta(n-k) \quad (8)$$

(7) reduces to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) = x(n) \quad (9)$$

Exercise 1: Verify (8)

If $x(n)$ is real, setting

$$X(\omega) = A(\omega)e^{j\theta(\omega)} \quad (10)$$

and making use of (5) to write

$$\begin{aligned}X(\omega)e^{j\omega n} + X(-\omega)e^{-j\omega n} &= X(\omega)e^{j\omega n} + \left[X(\omega)e^{j\omega n}\right]^* \\ &= 2\,\mathrm{Re}\left[X(\omega)e^{j\omega n}\right] \\ &= 2A(\omega)\cos(\omega n + \theta(\omega))\end{aligned} \quad (11)$$

we arrive at

$$x(n) = \frac{1}{\pi}\int_0^{\pi} A(\omega)\cos(\omega n + \phi(\omega))\,d\omega \quad (12)$$

This represents any real signal $x(n)$ as a superposition of (real) sinusoids with frequencies in the range $0\le\omega\le\pi$, $0\le f\le 1/2$.

Discrete-time and continuous-time Fourier transforms

If a discrete signal is a sampled version of a continuous signal

$$x(n) = x_a(nT_s) \quad (13)$$

there is an important relationship between their Fourier transforms. The continuous-time Fourier and inverse Fourier transforms are

$$X_a(\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt \quad (14)$$

and

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\Omega)e^{j\Omega t}d\Omega \quad (15)$$

Substituting the latter into (13) and using $T_s = 1/F_s$ we obtain

$$x(n) = x_a(n/F_s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\Omega) e^{j(\Omega/F_s)n} d\Omega \tag{16}$$

Since $F = F_s f$ and $\Omega = F_s \omega$ this becomes

$$x(n) = \frac{F_s}{2\pi}\int_{-\infty}^{\infty} X_a(F_s\omega) e^{j\omega n} d\omega \tag{17}$$

Comparing this to (6) we see that if

$$X_a(F_s\omega) = 0 \text{ for } |\omega| > \pi \tag{18}$$

then

$$X(\omega) = F_s X_a(F_s\omega) \tag{19}$$

and the continuous- and discrete-time Fourier transforms are (essentially) identical. However, if (18) is not true then we can write (17) as

$$x(n) = \frac{F_s}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} X_a(F_s\omega + k2\pi) e^{j\omega n} d\omega \tag{20}$$

Comparing this to (6) we have

$$X(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a(F_s[\omega + k2\pi]) \tag{21}$$

This is another way to express the sampling theorem. The $k \neq 0$ terms are frequency aliases. To avoid these we need to have F_s large enough so that (18) is true.