Lecture 5

The inverse z transform

Introduction

Given a signal $x(n)$ we can (in principle) calculate the *z* transform using the formula

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{1}$$

Let's consider the "inverse" problem where we start with $X(z)$ and want to calculate $x(n)$. There is a formal *inverse z transform* that can be used to do this. However, it requires the evaluation of a complex contour interval and is not of much practical use. (A similar statement applies to the inverse Laplace transform.) A practical method for calculating the inverse *z* transform is similar to the approach used to find the inverse Laplace transform; we use partial fractions to express $X(z)$ as a sum of simple terms, each of which can be inverse transformed by inspection.

Partial fraction expansion

Suppose we wish to calculate the inverse *z* transform of

$$X(z)=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}} \tag{2}$$

If $X(z)$ has only *simple poles* (denominator has no repeated roots), a systematic way to do so is as follows.

1. Multiply by $\frac{z^k}{z^k}$ where z^{-k} is the most negative power of *z* in either the numerator or denominator. The resulting rational function will contain no negative powers of *z*.
2. If the result is not a proper rational function (it is usually the ratio of two k^{th} order polynomials) divide it by *z* to get an expression for $\frac{X(z)}{z}$ that is proper.
3. Find all roots of the denominator. These are the poles $p_1, p_2, \ldots$ which can be real or complex. If they are complex they come in conjugate pairs.
4. For each pole add a term $\frac{A_i}{z-p_i}$ to the partial fraction expansion. If the pole is complex there will also be a conjugate $\frac{A_i^*}{z-p_i^*}$ term.
5. Solve for the constants A_i.

6. Multiply both sides of the equation by z to get $X(z)$ equal to a sum of terms such as $A_i \frac{z}{z-p_i}$.

7. Write $A_i \frac{z}{z-p_i} = \frac{A_i}{1-p_i z^{-1}}$ for all values of i. Replace each such term with the inverse z-transform $A_i p_i^n u_s(n)$.

Here's an algebraic example.

Example 1: If $X(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}$ what is $x(n)$?

Step 1. Multiply by $\frac{z}{z}$ to get $X(z) = \frac{b_0 z + b_1}{z + a_1}$. This is a rational function of z, but it is not a proper rational function (the order of the numerator is not less than the order of the denominator).

Step 2. Form the proper rational function $\frac{X(z)}{z} = \frac{b_0 z + b_1}{z(z+a_1)}$.

Step 3. The denominator has two roots: $0, -a_1$.

Step 4. Express $\frac{X(z)}{z}$ as partial fractions: $\frac{b_0 z + b_1}{z(z+a_1)} = \frac{A_1}{z+a_1} + \frac{A_2}{z}$.

Step 5. Clear fractions, $b_0 z + b_1 = A_1 z + A_2(z+a_1) = (A_1 + A_2) z + a_1 A_2$, and solve for the coefficients, $A_2 = \frac{b_1}{a_1}$, $A_1 + A_2 = b_0 \rightarrow A_1 = b_0 - \frac{b_1}{a_1}$.

Step 6. $X(z) = A_1 \frac{z}{z+a_1} + A_2$

Step 7. $X(z) = A_1 \frac{1}{1+a_1 z^{-1}} + A_2$ so, by inspection,

$$x(n) = A_1 (-a_1)^n u_s(n) + A_2 \delta(n) = \left(b_0 - \frac{b_1}{a_1}\right)(-a_1)^n u_s(n) + \frac{b_1}{a_1}\delta(n)$$

Now let's look at a numerical example.

Example 2: If $X(z) = \frac{1 + \frac{1}{2} z^{-1} + z^{-2}}{1 + \frac{3}{8} z^{-1} + \frac{9}{16} z^{-2}}$ what is $x(n)$?

Step 1. Multiply by $\frac{z^2}{z^2}$ to get $X(z)=\dfrac{z^2+\frac{1}{2}z+1}{z^2+\frac{3}{8}z+\frac{9}{16}}$.

Step 2. $\dfrac{X(z)}{z}=\dfrac{z^2+\frac{1}{2}z+1}{z\left(z^2+\frac{3}{8}z+\frac{9}{16}\right)}$ is a proper rational function.

Step 3. The denominator has three roots: 0 and $-\frac{3}{16}\pm j\frac{3}{16}\sqrt{15}$. Call these last two conjugate roots p and p^*.

Step 4. $\dfrac{z^2+\frac{1}{2}z+1}{z(z^2+\frac{3}{8}z+\frac{9}{16})}=\dfrac{A_1}{z}+\dfrac{A_2}{z-p}+\dfrac{A_2^*}{z-p^*}$

Step 5. Multiply through by $z(z^2+\frac{3}{8}z+\frac{9}{16})=z(z-p)(z-p^*)$ to get

$$z^2+\frac{1}{2}z+1=A_1(z^2+\frac{3}{8}z+\frac{9}{16})+A_2z(z-p^*)+A_2^*z(z-p)$$

Setting $z=0$ we find $1=A_1\frac{9}{16}\rightarrow A_1=\frac{16}{9}=1.778$.

Setting $z=p$ we have $p^2+\frac{p}{2}+1=A_2p(p-p^*)\rightarrow A_2=\dfrac{p^2+\frac{1}{2}p+1}{p(p-p^*)}$. Plugging in the value of p we get $A_2=-\frac{7}{18}+j\frac{1}{18\sqrt{15}}=-0.3889+j0.01434$.

Step 6. $X(z)=A_1+A_2\dfrac{z}{z-p}+A_2^*\dfrac{z}{z-p^*}$

Step 7. $X(z)=A_1+A_2\dfrac{1}{1-pz^{-1}}+A_2^*\dfrac{1}{1-p^*z^{-1}}$ so

$$\begin{aligned}x(n)&=A_1\delta(n)+A_2p^nu_s(n)+A_2^*(p^*)^nu_s(n)\\&=A_1\delta(n)+2\,\mathrm{Re}\{A_2p^n\}u_s(n)\end{aligned}$$

(because the sum of a complex number and its conjugate is twice the real part of the number). This is most conveniently manipulated in polar format

$$A_2=-\frac{7}{18}+j\frac{1}{18\sqrt{15}}=-0.3889+j0.01434=0.3892e^{j3.105}$$

$$p=-\frac{3}{16}+j\frac{3}{16}\sqrt{15}=-0.1875+j0.7262=0.75e^{j1.823}$$

$$\mathrm{Re}\{A_2 p^n\}=\mathrm{Re}\{0.3892e^{j3.105}0.75^n e^{j1.823n}\}=0.3892(0.75^n)\cos(1.823n+3.105)$$

so

$$x(n)=1.78\delta(n)+0.778(0.75^n)\cos(1.82n+3.10)$$

Try this for yourself

Exercise 1: If $X(z)=\dfrac{4-\frac{7}{4}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$ what is $x(n)$?

Answer: $\left[3\left(\frac{1}{4}\right)^n+\left(\frac{1}{2}\right)^n\right]u_s(n)$

Repeated roots

As it is for the Laplace transform, it is possible to have repeated roots of the rational function denominator of the z transform. We will not make much use of this idea, but we cover it for completeness. Suppose we find

$$\frac{X(z)}{z}=\frac{az+b}{(z-p)^2} \tag{3}$$

which has a double root at $z=p$. The partial-fraction expansion is

$$\frac{az+b}{(z-p)^2}=\frac{A}{z-p}+\frac{B}{(z-p)^2} \tag{4}$$

Clearing fractions

$$az+b=A(z-p)+B \tag{5}$$

and equating the z coefficients and constant terms we solve for the coefficients

$$A=a \ , \ b=B-pA\rightarrow B=b+pa \tag{6}$$

Multiplying through by z

$$\begin{aligned} X(z)&=A\frac{z}{z-p}+B\frac{z}{(z-p)^2} \\ &=A\frac{1}{1-pz^{-1}}+B\frac{z^{-1}}{(1-pz^{-1})^2} \end{aligned} \tag{7}$$

Using our z-transform table we have

$$x(n)=[a p^n+(b+pa)n p^{n-1}]u_s(n) \tag{8}$$

Another point of view is to treat the double root as two distinct roots at $p, p-\epsilon$ followed by letting $\epsilon\rightarrow 0$. We write

$$X(z)=\frac{az+b}{(z-p)(z-p+\epsilon)}=\frac{A}{z-p}+\frac{B}{z-p+\epsilon} \tag{9}$$

The inverse transform gives us

$$x(n)=[Ap^n+B(p-\epsilon)^n]u_s(n) \tag{10}$$

Clearing fractions in (9)

$$az+b=A(z-p+\epsilon)+B(z-p) \tag{11}$$

from which

$$a=A+B \text{ and } b=-p(A+B)+\epsilon A \tag{12}$$

The solution can be written

$$A=\frac{b+pa}{\epsilon}\ ,\ B=a-A \tag{13}$$

Now consider

$$\begin{aligned} Ap^n+B(p-\epsilon)^n &= Ap^n+(a-A)(p-\epsilon)^n \\ &= A\left[p^n-(p-\epsilon)^n\right]+a(p-\epsilon)^n \\ &= \frac{b+pa}{\epsilon}\left[p^n-(p-\epsilon)^n\right]+a(p-\epsilon)^n \end{aligned} \tag{14}$$

Since $(p-\epsilon)^n=p^n-n\epsilon p^{n-1}+\cdots$, in the $\epsilon\rightarrow 0$ limit this becomes

$$\frac{b+pa}{\epsilon}\left[n\epsilon p^{n-1}\right]+a(p-\epsilon)^n\rightarrow(b+pa)np^{n-1}+ap^n \tag{15}$$

and

$$x(n)=\left[ap^n+(b+pa)np^{n-1}\right]u_s(n) \tag{16}$$

Analyzing LTI systems

The *z* and inverse-*z* transforms allow us to explicitly calculate the output of an LTI system. If the transfer function is $H(z)$ and the input is $x(n)$, we calculate $X(z)$ and then set

$$Y(z)=H(z)X(z) \tag{17}$$

Finally we inverse transform $Y(z)$ to obtain $y(n)$. Let's illustrate with an example.

Example 3: A filter has transfer function $H(z)=\frac{1}{1-0.5z^{-1}}$. The signal $x(n)=(-0.7)^n u_s(n)$ is input. What is the output $y(n)$?

The z transform of the input is $X(z)=\frac{1}{1+0.7z^{-1}}$, so the z transform of the output is

$$\begin{aligned} Y(z) &= \frac{1}{1-0.5z^{-1}}\frac{1}{1+0.7z^{-1}} \\ &= \frac{z}{z-0.5}\frac{z}{z+0.7} \end{aligned}$$

and $$\frac{Y(z)}{z}=\frac{z}{(z-0.5)(z+0.7)}=\frac{A}{z-0.5}+\frac{B}{z+0.7}$$

Clearing fractions $z=A(z+0.7)+B(z-0.5)$ gives us the equations $1=A+B$, $0=0.7A-0.5B\rightarrow B=1.4A$, so $1=2.4A\rightarrow A=0.4167$, $B=0.5833$. Since

$$Y(z)=A\frac{1}{1-0.5z^{-1}}+B\frac{1}{1+0.7z^{-1}}$$

we have $$y(n)=\left[0.417\cdot 0.5^n+0.583(-0.7)^n\right]u_s(n)$$

Let's try with a different input.

Example 4: Repeat the previous example with $x(n)=\sin(0.1\pi n)u_s(n)$ as input.

The *z* transform of the input is

$$X(z)=\frac{\sin(0.1\pi)z^{-1}}{1-2\cos(0.1\pi)z^{-1}+z^{-2}}=\frac{0.3090z^{-1}}{1-0.6180z^{-1}+z^{-2}}$$

Therefore, the *z* transform of the output is

$$Y(z)=\frac{1}{1-0.5z^{-1}}\frac{0.3090z^{-1}}{1-0.6180z^{-1}+z^{-2}}$$

and $$\frac{Y(z)}{z}=\frac{1}{z-0.5}\frac{0.309z}{z^2-0.618z+1}$$

This has poles at $p_1=0.5$, $p_2=(0.618+\sqrt{0.618^2-4})/2=0.309+j0.9511$, and p_2^*. The partial-fraction expansion is

$$\frac{Y(z)}{z}=\frac{A}{z-p_1}+\frac{B}{z-p_2}+\frac{B^*}{z-p_2^*}$$

Clearing fractions

$$0.309z=A(z-p_2)(z-p_2^*)+B(z-p_1)(z-p_2^*)+B^*(z-p_1)(z-p_2)$$

Let $z=p_1$ to get $0.309p_1=A(p_1-p_2)(p_1-p_2^*)$ from which

$$A=\frac{0.309p_1}{(p_1-p_2)(p_1-p_2^*)}=0.164$$

Let $z=p_2$ to get $0.309p_2=B(p_2-p_1)(p_2-p_2^*)$ from which

$$B=\frac{0.309p_2}{(p_2-p_1)(p_2-p_2^*)}=-0.08209-j0.1460=0.167\angle-2.083\,\text{rad}$$

Since
$$Y(z)=A\frac{z}{z-p_1}+B\frac{z}{z-p_2}+B^*\frac{z}{z-p_2^*}$$
$$=A\frac{1}{1-p_1z^{-1}}+B\frac{1}{1-p_2z^{-1}}+B^*\frac{1}{1-p_2^*z^{-1}}$$

we have

$$y(n) = \left[A\, p_1^n + 2\,\mathrm{Re}\left\{ B\, p_2^n \right\} \right] u_s(n)$$
$$= \left[A\, p_1^n + 2\,|B|\,|p_2|^n \cos(\omega n + \phi) \right] u_s(n)$$

with $\omega = \angle p_2$ and $\phi = \angle B$. Since $p_2 = 1 \angle 1.256\,\mathrm{rad}$

$$y(n) = \left[0.164 \cdot 0.5^n + 0.335 \cos(1.256\, n - 2.083) \right] u_s(n)$$

Here is how some of these calculations can be done in Scilab.

```
deff('u=arg(v)','u=atan(imag(v),real(v))');
p1 = 0.5;
p2 = (0.618+sqrt(0.618^2-4))/2;
A = 0.309*p1/((p1-p2)*(p1-conj(p2)));
B = 0.309*p2/((p2-p1)*(p2-conj(p2)));
mprintf("A = %f\n", A);
mprintf("B = %f < %f\n", abs(B), arg(B));
mprintf("p2 = %f < %f\n", abs(p2), arg(p2));
```

The output is

```
A = 0.164187
B = 0.167465 < -2.083129
p2 = 1.000000 < 1.256655
```