Lecture 1

Review of linear systems

Introduction

Most of the digital signal processing concepts we will study are discrete forms of the continuous linear-systems concepts you studied in EE 261, 321, and 341. Accordingly we begin with a brief review of those ideas.

One-dimensional continuous signals

A one-dimensional (1D) continuous *signal* $x(t)$ is a function that assigns a value of the *dependent variable* x to every value of the *independent variable* t. We will also call this an *analog signal*. In circuit analysis the independent variable t is time and the dependent variable x is typically a voltage or current at some point of the circuit. In mechanics x might be a linear position or an angle. We will focus on audio signals and treat x as a dimensionless amplitude.

Unless otherwise stated we assume t varies over all possible values $-\infty < t < \infty$. For example, the most general sinusoidal signal takes the form

$$x(t) = A\cos(\Omega t + \phi) \tag{1}$$

where $\Omega = 2\pi F$ is the *angular frequency* (radians per second), F is the *frequency* (Hz or cycles per second), A is the *amplitude*, and ϕ is the *phase* (radians, or sometimes expressed in degrees).

The following are some important functions. We will use the discrete forms of these extensively.

Unit-step function

The unit-step function can be used to represent a signal that "turns on" at $t=0$.

$$u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{2}$$

Multiplying another signal by the unit-step function models that signal turning on at $t=0$. A sinusoid turning on is

$$x(t) = A\cos(\Omega t + \phi)\, u_s(t) \tag{3}$$

We can use a difference of unit-step functions to model a signal that turns on and then off.

$$p(t) = u_s(t - t_{\text{on}}) - u_s(t - t_{\text{off}}) \tag{4}$$

is rectangular pulse that turns on, off at $t_{\text{on}}, t_{\text{off}}$. We can extend this idea to non-rectangular pulses. For example, the function

$$x(t) = x_{\text{on}}\, e^{-(t-t_{\text{on}})/\tau}\left[u_s(t-t_{\text{on}}) - u_s(t-t_{\text{off}})\right] \tag{5}$$

describes an exponential that begins at t_{on}, decays with time constant τ, and ends at t_{off}.

Rectangle function

A more compact way to represent on/off behavior is the *rect* ("rectangle") function

$$\mathrm{rect}(t)=u_s(t+1/2)-u_s(t-1/2)=\begin{cases}1 & |t|\le\frac{1}{2}\\ 0 & |t|>\frac{1}{2}\end{cases} \tag{6}$$

This is a pulse of width $\Delta t=1$ centered on $t=0$. The function $A\,\mathrm{rect}\big((t-t_0)/T\big)$ therefore represents a rectangular pulse of amplitude A and width T centered at $t=t_0$.

Delta function

The delta function is the limiting case of a rectangular pulse of zero width and infinite amplitude.

$$\delta(t)=\lim_{w\to 0}\frac{1}{w}\mathrm{rect}\left(\frac{t}{w}\right) \tag{7}$$

The delta function is also called an *impulse*. An important application is its use to represent sampling

$$x(t)=\int_{-\infty}^{\infty}\delta(t-\tau)x(\tau)d\tau=\lim_{w\to 0}\frac{1}{w}\int_{t-w/2}^{t+w/2}x(\tau)d\tau \tag{8}$$

Sinc function

The sinc function (pronounced "sink") is

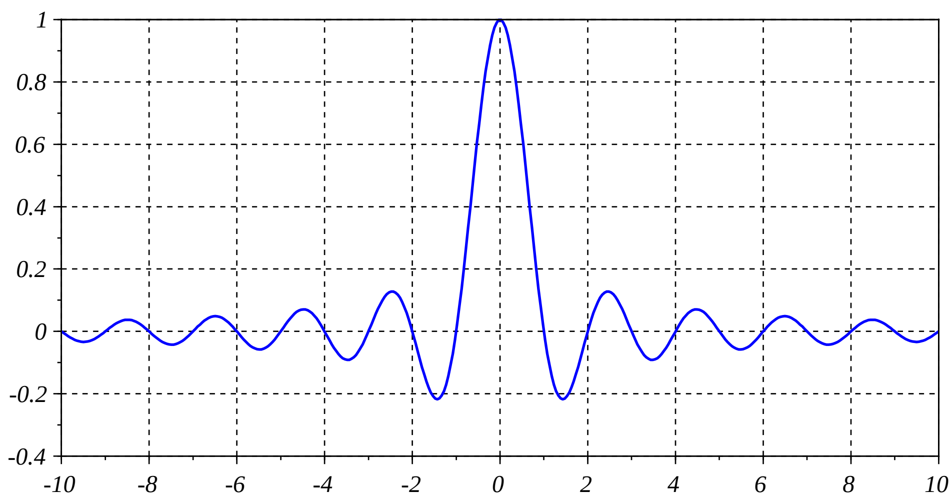

Fig. 1 The sinc function.

$$\operatorname{sinc}(t) \stackrel{\text{def}}{=} \frac{\sin(\pi t)}{\pi t} \tag{9}$$

and is plotted in Fig. 1. It is unity at $t=0$ and zero for non-zero integer values of *t*. The *main-lobe* extends over $-1 \le t \le 1$. Smaller *sidelobe* ripples characterize $|t| > 1$. As we will see, the sinc function plays a central role in filter theory.

Sinusoids

The most general sinusoid is (1). For analytic purposes it can be very useful to represent sinusoids in terms of complex exponentials. Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{10}$$

allows us to write

$$\begin{aligned} \cos\theta &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\ \sin\theta &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \end{aligned} \tag{11}$$

We can use these to express a general sinusoid as

$$A\cos(\Omega t + \phi) = \operatorname{Re}\left\{A e^{j\phi} e^{j\Omega t}\right\} = \frac{1}{2}\left(A e^{j\phi} e^{j\Omega t} + A e^{-j\phi} e^{-j\Omega t}\right) \tag{12}$$

SISO systems

A *single-input, single-output* (SISO) continuous system (Fig. 2) transforms an input signal $x(t)$ into an output signal $y(t)$.

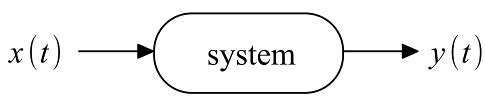

Fig. 2: Continuous SISO system.

We write

$$T[x(t)] = y(t) \tag{13}$$

to represent the *transformation* that produces *y* from *x*. There is not much we can say about a transformation unless we somehow constrain its properties.

Linear systems

A *linear system* satisfies the principle of *superposition*. If $T[x_1(t)] = y_1(t)$ and $T[x_2(t)] = y_2(t)$ then for any constants a_1, a_2 we have

$$T[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t) \tag{14}$$

If a system is linear then each output value $y(t)$ is a linear combination of the input values $x(t)$, We can express this as

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau) x(\tau) d\tau \tag{15}$$

If our input is an impulse at time τ, the output is

$$h(t,\tau)=\int_{-\infty}^{\infty} h(t,u)\delta(u-\tau)\,du \tag{16}$$

We call $h(t,\tau)$ the *impulse response* of the system. Note that for a general linear system $h(t,\tau)$ is a 2D function. This means that the time response (*t*) of the system to impulses at different times (τ) can be arbitrarily related. That is, the "behavior" of the system can change through time. As an example, it might behave like a lowpass filter now and later behave like a highpass filter in response to time-varying filter parameters, for example, someone turning knobs on an audio equalizer.

Linear time-invariant systems

A linear system is *time-invariant* if it "behaves" the same at all times, and we call it a *Linear Time-Invariant* (LTI) system. Shifting an input in time gives the same output, merely with a corresponding time shift. If $T[x(t)]=y(t)$ then

$$T[x(t-t_0)]=y(t-t_0) \tag{17}$$

If we define the response to an impulse at time 0 as

$$h(t)\overset{\text{def}}{=}h(t,0) \tag{18}$$

then the response to an impulse at time τ is simply the shifted version

$$h(t-\tau)\overset{\text{def}}{=}h(t,\tau) \tag{19}$$

The impulse response is now a 1D function. The input-output relation (15) reduces to

$$y(t)=\int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau \tag{20}$$

which is called the *convolution* of $h(t)$ and $x(t)$. A change of variable $u=t-\tau$ allows us to write

$$y(t)=\int_{-\infty}^{\infty} h(u)x(t-u)\,du \tag{21}$$

We use an asterisk operator

$$y(t)=h(t)*x(t)=x(t)*h(t) \tag{22}$$

to denote convolution.

Causal systems

A *causal system* is one in which the present does not depend on the future. All *realtime* systems must be causal since we live in a causal universe. An LTI system is causal if and only if

$$h(t)=0\,,\, t<0 \tag{23}$$

This says there can be no impulse response output before the impulse input occurs.

Not all systems are causal. A non-causal system is possible if "the future has already occurred." As an example, a complete piece of music has been recorded and you wish to filter it. At any place in the signal the future signal samples are already known so the current output can "depend on the future." In "near realtime" systems this can be implemented by introducing a time delay between input and output. This idea is used extensively for DSP processing of voice signals in cell phone and internet applications. The delay is typically a few tens of milliseconds, which is not enough to have a noticeable effect on a conversation.

Fourier series

A periodic function $x(t \pm T) = x(t)$ can be represented by a *Fourier series*

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right] \tag{24}$$

where $\omega_0 = (2\pi)/T$. The coefficients are

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt \tag{25}$$

and

$$\begin{aligned} a_n &= \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t)\, dt \\ b_n &= \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t)\, dt \end{aligned} \tag{26}$$

If a function $x(t)$ is not periodic the series (24) can still be used to represent it over the interval $0 \le t \le T$.

Using the trigonometric identity

$$\cos(u - v) = \cos(u)\cos(v) + \sin(u)\sin(v)$$

we can express a Fourier series in the form

$$x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega_0 t - \theta_n) \tag{27}$$

with

$$\begin{aligned} a_n &= A_n \cos(\theta_n) \\ b_n &= A_n \sin(\theta_n) \end{aligned} \tag{28}$$

Another useful form is

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{29}$$

with

$$c_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t}\, dt \tag{30}$$

If $x(t)$ is a real signal, then for $n \ge 0$, $c_n = A_n e^{-j\theta_n}$ and $c_{-n} = c_n^*$.

Laplace transform

The Laplace transform derives its usefulness from the fact that a homogeneous, ordinary differential equation with constant coefficients, such as

$$\ddot{x}+b\,\dot{x}+c\,x=0 \tag{31}$$

has solutions of the form

$$x(t)=a\,e^{st} \tag{32}$$

Since $\frac{d}{dt}x^{st}=s\,e^{st}$, plugging (32) into (31) results in

$$\left(s^2+bs+c\right)e^{st}=0 \tag{33}$$

Since e^{st} is never zero, this requires

$$s^2+bs+c=0 \tag{34}$$

We have converted an n^{th} order ODE into an n^{th} order polynomial.

The "two-sided" *Laplace transform* of $x(t)$ is defined as

$$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}\,dt \tag{35}$$

where $s=\sigma+j\omega$ is an arbitrary complex number. If $x(t)=0\,,t<0$ this becomes the "normal" Laplace transform

$$X(s)=\int_{0}^{\infty} x(t)e^{-st}\,dt \tag{36}$$

That is most often used in applications. The Laplace transform of a unit step $u_s(t)$ is

$$X(s)=\int_{0}^{\infty} e^{-st}\,dt=\left.\frac{e^{-st}}{-s}\right|_0^\infty=\frac{1}{s} \tag{37}$$

The transform of $x(t)=e^{-at}u_s(t)$ is

$$X(s)=\int_{0}^{\infty} e^{-at}e^{-st}\,dt=\left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_0^\infty=\frac{1}{s+a} \tag{38}$$

Given a Laplace transform, the time-domain function can be calculated using a formal *inverse Laplace transform*

$$x(t)=\frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds \tag{39}$$

where σ must be appropriately chosen so the integral converges. However, in practice the inverse Laplace transform is calculated using a partial fraction expansion (reviewed below) and (38).

Fourier transform

Substituting $s=j\omega$ in (35) we obtain the *Fourier transform*

$$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \tag{40}$$

The *inverse Fourier transform* is

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega \tag{41}$$

This represents $x(t)$ as a superposition of complex exponentials $e^{j\omega t}$. It's analogous to the Fourier series (29), but can represent arbitrary non-periodic functions.

Convolution theorem

Let $y(t)=h(t)*x(t)$, that is

$$y(t)=\int_{-\infty}^{\infty}h(t-\tau)x(\tau)d\tau \tag{42}$$

The Laplace transform of $y(t)$ is

$$Y(s)=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}h(t-\tau)x(\tau)d\tau\right]e^{-st}dt \tag{43}$$

Let $u=t-\tau$, $du=dt$. Then the above expression becomes

$$Y(s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(u)x(\tau)e^{-s(u+\tau)}d\tau\, du=\int_{-\infty}^{\infty}h(u)e^{-su}du\int_{-\infty}^{\infty}x(\tau)e^{-s\tau}d\tau \tag{44}$$

Therefore

$$Y(s)=H(s)X(s) \tag{45}$$

This is the *convolution theorem*. The Laplace transform of a convolution is the product of Laplace transforms. With $s=j\omega$ we obtain the Fourier transform version

$$Y(j\omega)=H(j\omega)X(j\omega) \tag{46}$$

Partial fraction expansion

In applications of the Laplace transform to LTI systems, we typically deal with transforms that are rational functions of *s* (ratio of polynomials in *s*). Let the M^{th} order numerator polynomial of $H(s)$ be $b(s)$ with roots $z_1, z_2, \ldots, z_M$. These roots are the *zeros* of $H(s)$. Let the N^{th} order denominator polynomial be $a(s)$ with roots $p_1, p_2, \ldots, p_N$. These are the *poles* of $H(s)$. We assume roots can be real or complex. Complex roots come in conjugate pairs. For simplicity we will assume there are no repeated roots. We write

$$H(s)=\frac{b(s)}{a(s)}=\frac{b_0(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)} \tag{47}$$

We assume $H(s)$ is a *proper rational function* with $N>M$. Then we can express it as a *partial fraction expansion* of *N* terms, one for each pole

$$\frac{b_0(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}=\frac{A_1}{s-p_1}+\frac{A_2}{s-p_2}+\cdots+\frac{A_N}{s-p_N} \tag{48}$$

Using (38) we have

$$h(t)=\left[A_1e^{p_1t}+A_2e^{p_2t}+\cdots A_Ne^{p_Nt}\right]u_s(t) \tag{49}$$

For $h(t)$ to not "blow up" at $t=\infty$, all real poles must be non-positive ($p_i \le 0$). For a complex pole $p_i = \sigma_i + j\omega_i$ we get a term

$$A_i e^{\sigma_i t} e^{j\omega_i t} \tag{50}$$

and we require $\sigma_i \le 0$. In general the real part of every pole must be non-positive for $h(t)$ to remain finite as $t \to \infty$.

Note that for any complex pole we will also have a conjugate term

$$A_i^* e^{\sigma_i t} e^{-j\omega_i t} \tag{51}$$

The sum of these is real

$$2 e^{\sigma_i t} \operatorname{Re}\left[A_i e^{j\omega_i t}\right] = 2|A_i| e^{\sigma_i t} \cos(\omega_i t + \theta_i) \tag{52}$$

where θ_i is the argument of A_i.

A few useful Laplace transform pairs

$x(t)$	$X(s)$
$\delta(t)$	1
$u_s(t)$	$\frac{1}{s}$
$t^n u_s(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at} u_s(t)$	$\frac{1}{s+a}$