Equalization

Introduction

We have spent considerable time discussing the problems that arise from multi-path channels.

Theory

As we've discussed previously, a wireless channel can be characterized by an impulse response or transfer function. If $x(t)$ is a transmitted signal, the received signal $y(t)$ is related to this by

$$\begin{gathered} y(t) = h(t) * x(t) \\ Y(f) = H(f)X(f) \end{gathered} \tag{33.1}$$

where * denotes convolution. If we knew what $H(f)$ is we could in principle multiply $Y(f)$ by the inverse of $H(f)$ to recover $X(f)$. In the time domain this would correspond to a convolution. That is, it should be possible to implement

$$\begin{gathered} X(f) = H^{-1}(f)Y(f) \\ x(t) = h_{eq}(t) * y(t) \end{gathered} \tag{33.2}$$

This process is referred to as *equalization*. More generally it falls under the topic of *inverse filtering*, or *deconvolution.* The relations between the time and frequency domains are

$$\begin{gathered} h(t) = \int H(f)e^{j2\pi ft}df \\ h_{eq}(t) = \int H^{-1}(f)e^{j2\pi ft}df \end{gathered} \tag{33.3}$$

We seek the condition

$$x(t) = h_{eq}(t) * h(t) * x(t) \tag{33.4}$$

in other words

$$h_{eq}(t) * h(t) = \delta(t) \tag{33.5}$$

If we can find the required $h_{eq}(t)$ then we can undo the effects of the multipath channel.

Implementation

Allowing for the fact that the equalizer may not work perfectly, let's write

$$\hat{x}(t) = \int_0^\infty h_{eq}(\tau)y(t-\tau)d\tau \tag{33.6}$$

where $\hat{x}(t)$ hopefully is close to $x(t)$. To implement the operations in the discrete (sampled) domain, let

$$h_{eq}(t) = \sum_{k=0}^{N} w_k \delta(t - k\Delta t) \tag{33.7}$$

then

$$\hat{x}(n\Delta t) = \sum_{k=0}^{N} w_k y(n\Delta t - k\Delta t) \tag{33.8}$$

and we can write

$$\hat{x}_n = \sum_{k=0}^{N} w_k y_{n-k} \tag{33.9}$$

In this manner we define the equalizer impulse response in terms of $N+1$ *weights* w_k. Our problem is to find those weights that make the equalizer output as close to the transmitted signal as possible.

Note that

$$\begin{aligned} H_{eq}(f) &= \int_0^{\infty} h_{eq}(t) e^{-j2\pi f t} dt \\ &= \sum_{k=0}^{N} w_k e^{-j2\pi f k\Delta t} \end{aligned} \tag{33.10}$$

so this approach is equivalent to approximating $H^{-1}(f)$ by $N+1$ sinusoids. Let's use the vector notation

$$\hat{x}_n = \mathbf{w}^T \mathbf{y}_n \quad ; \quad \mathbf{w} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{bmatrix}, \ \mathbf{y}_n = \begin{bmatrix} y_n \\ y_{n-1} \\ \vdots \\ y_{n-N} \end{bmatrix} \tag{33.11}$$

Then the equalizer error is

$$e_n = x_n - \hat{x}_n \tag{33.12}$$

We want this error to be as small as possible. We calculate

$$\begin{aligned} \left\langle e_n{}^2 \right\rangle &= \left\langle x_n{}^2 \right\rangle + \left\langle \hat{x}_n{}^2 \right\rangle - 2\left\langle x_n \hat{x}_n \right\rangle \\ &= \left\langle x_n{}^2 \right\rangle + \mathbf{w}^T \left\langle \mathbf{y}_n \mathbf{y}_n{}^T \right\rangle \mathbf{w} - 2\left\langle \mathbf{x}_n \mathbf{y}_n{}^T \right\rangle \mathbf{w} \end{aligned} \tag{33.13}$$

and look for the *weight vector* **w** that minimizes this quantity. It is convenient to use the following additional matrix-vector notation. A correlation matrix is defined as

$$\mathbf{R} = \left\langle \mathbf{y}_n \mathbf{y}_n^T \right\rangle = \begin{bmatrix} \left\langle y_n^2 \right\rangle & \left\langle y_n y_{n-1} \right\rangle & \cdots & \left\langle y_n y_{n-N} \right\rangle \\ \left\langle y_{n-1} y_n \right\rangle & \left\langle y_{n-1}^2 \right\rangle & \cdots & \left\langle y_{n-1} y_{n-N} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle y_{n-N} y_n \right\rangle & \left\langle y_{n-N} y_{n-2} \right\rangle & \cdots & \left\langle y_{n-N}^2 \right\rangle \end{bmatrix} \tag{33.14}$$

and a cross-correlation vector is defined as

$$\mathbf{p} = \left\langle \mathbf{x}_n \mathbf{y}_n \right\rangle = \begin{bmatrix} \left\langle x_n y_n \right\rangle \\ \left\langle x_n y_{n-1} \right\rangle \\ \vdots \\ \left\langle x_n y_{n-N} \right\rangle \end{bmatrix} \tag{33.15}$$

Now (33.13) can be written

$$\left\langle e_n^2 \right\rangle = \left\langle x_n^2 \right\rangle + \mathbf{w}^T \mathbf{R} \mathbf{w} - 2\mathbf{p}^T \mathbf{w} \tag{33.16}$$

This is a quadratic form in **w**, and standard linear algebra techniques give the solution for the minimum as

$$\mathbf{w} = \mathbf{R}^{-1} \mathbf{p} \tag{33.17}$$

We can then apply (33.11) to calculate the equalizer output $\hat{x}_n$.

Training Sequences

The above steps might seem a bit circular. In order to calculate the weight vector **w** using (33.17), we need to know the transmitted sequence x_n. We then use **w** to calculate $\hat{x}_n$ that approximates something we already know, namely x_n. Why this is useful is that we can perform this exercise for a known transmitted sequence, a *training sequence*, and then apply the resulting equalization filter to unknown data sequences. This will work provided the impulse response of the channel doesn't change appreciably. In practice, therefore, we would want to periodically send known training sequences interspersed with our data so that we could track changes in the channel impulse response.

An example of this is the GSM system that is the digital cellular standard in Europe. During a normal data burst, 116 bits of data have 26 training bits inserted in the middle to enable equalization.

Simulation

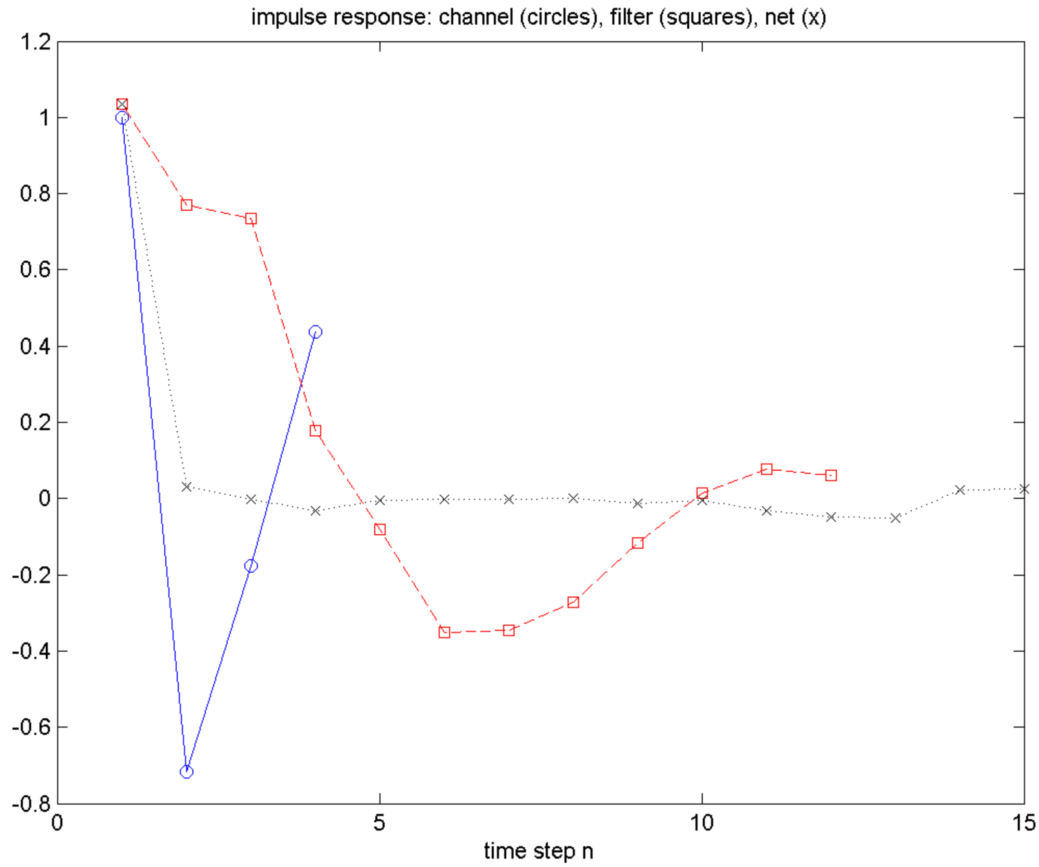

Figure 33.1: Equalization example. Circles show simulated channel impulse response. Squares show equalizer impulse response when adjusted to give minimum error for a known data sequence. X's show convolution of the two, which approximates a delta-function, as expected.

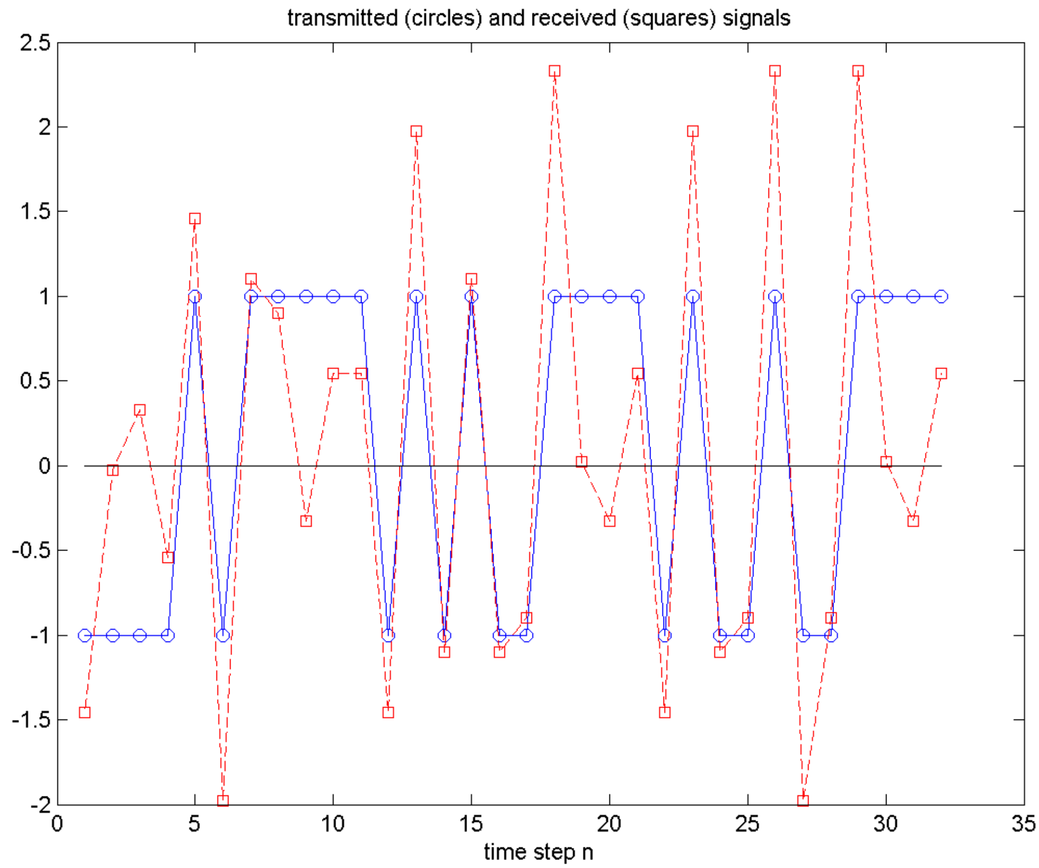

Figure 33.2: Simulated transmitted sequence (circles) and received sequence (squares) for channel impulse response shown in Fig. 33.1. In this case there are four bit errors in the 32-bit sequence.

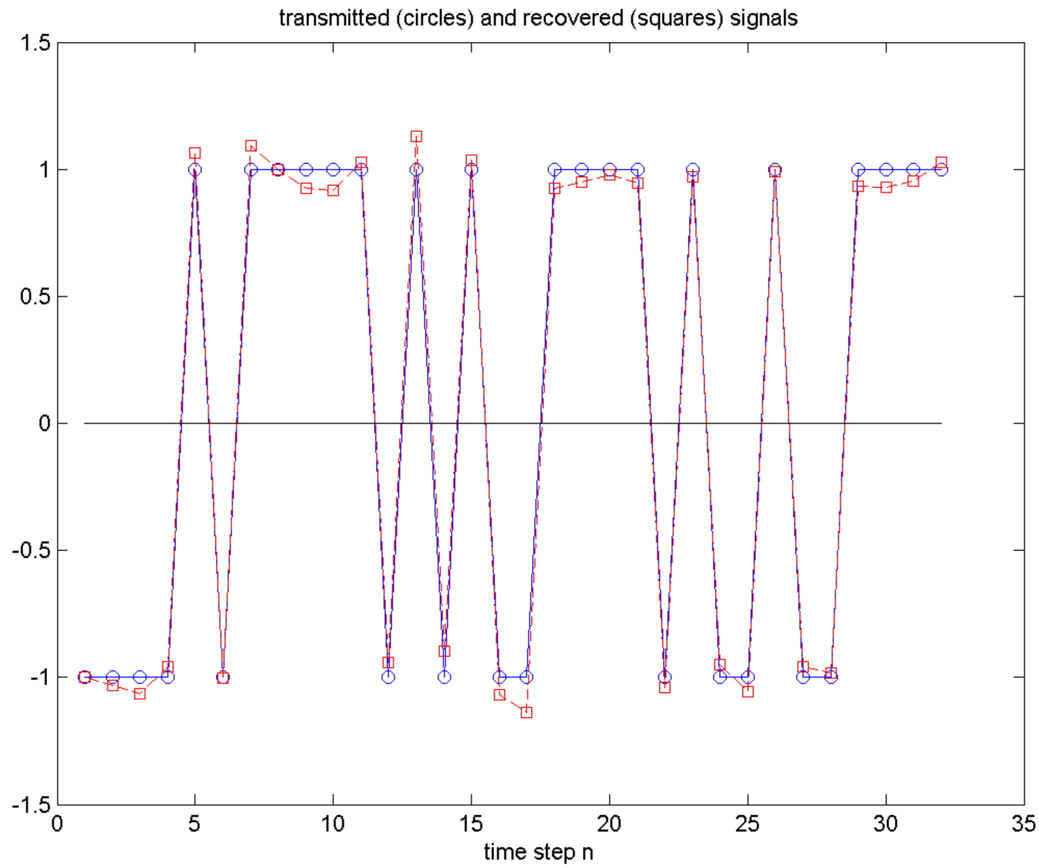

Figure 33.3: Simulated transmitted sequence (circles) and equalizer output sequence (squares) for channel and equalizer impulse response shown in Fig. 33.1. All bit errors have been corrected.

References

1. Rappaport, T. S., *Wireless Communications: Principles and Practice*, Prentice Hall, 1996, ISBN 0-13-375536-3.