Lecture 9

Motion Effects and Diversity Reception

1 Doppler Effect

When a transmitter and receiver are in relative motion, the received frequency can differ from the transmitted frequency. This is known as the *Doppler Effect*. Assume the time dependence of the transmitted signal is $e^{j2\pi f t}$ and the distance to the receiver is *r*. Then the time dependence of the received signal is $e^{j2\pi f(t-r/c)}$. If transmitter and/or the receiver are moving then *r* might vary with time. Let's write $r=r_0+\dot{r}t$. The time dependence of the received signal is then

$$\begin{aligned} e^{j2\pi f(t-r_0/c-\dot{r}t/c)} &= e^{j2\pi(f-f\dot{r}/c)t}e^{-j2\pi f r_0/c} \\ &= e^{j2\pi f' t}e^{-j2\pi f r_0/c} \end{aligned} \tag{1}$$

The received frequency is $f'=f+f_d$ where the *Doppler frequency* (or *Doppler shift*) is

$$f_d=-\frac{f\dot{r}}{c}=-\frac{\dot{r}}{\lambda} \tag{2}$$

and we've used $f\lambda=c$. The Doppler shift in Hz is equal to the relative velocity in wavelengths, e.g., a velocity of one wavelength per second produces a Doppler shift of one Hz. If the transmitter and receiver are moving apart the shift is negative; if they are moving closer the shift is positive.

2 Rayleigh Fading For a Moving Receiver

We have seen that Rayleigh fading arises when the field near the receiver is composed of a large number of plane wave components traveling in random directions. Fig. 1 shows an example 2D field distribution.

Let's consider what happens if a receiver is moving through such a distribution. Let's assume we are traveling down the *x* axis at some speed *v*. There are *N* plane waves with random amplitudes, phases, and propagation directions. The received voltage at position *x* will therefore be

$$V(x)=\sum_{k=1}^{N} a_k e^{j\phi_k} e^{-j\frac{2\pi}{\lambda}\cos\theta_k x} \tag{3}$$

where θ_k is the angle between the *x*-axis and the direction of propagation of the k^{th} plane wave. Since $x=vt$ we can write the voltage as a function of time

$$V(x)=\sum_{k=1}^{N} a_k e^{j\phi_k} e^{-j\frac{2\pi}{\lambda}v\cos\theta_k t} \tag{4}$$

This is a sum of sinusoids. The maximum frequency present in this signal occurs for $\theta=0,\pi$ and is $f_{\max}=v/\lambda$. The Sampling Theorem says we can capture all the information in such a signal if we sample at the rate $f_s=2f_{\max}=2v/\lambda$. The sampling period is therefore $T_s=\lambda/(2v)$, and the distance the receiver travels during this time is $\Delta x=vT_s=\lambda/2$. An example simulation is given in Fig. 2. The received signal varies over a large range of amplitudes in a very erratic fashion. Our receiver needs to be able to function under such extreme conditions.

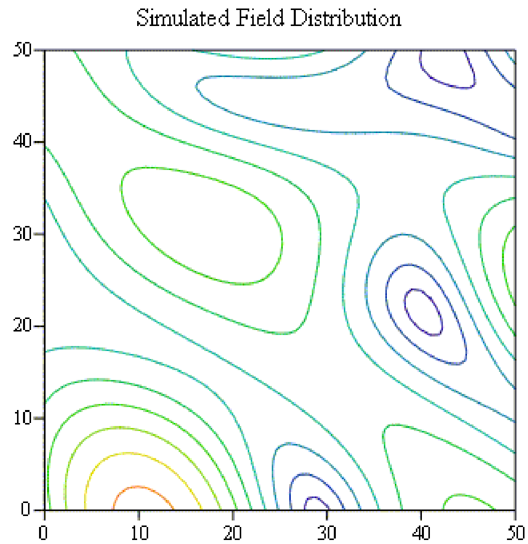

Fig. 1 Simulated 2D field distribution.

It is convenient to normalize the voltage to get the dimensionless quantity $\rho = V/V_{rms}$ where $V_{rms} = \sqrt{2}\sigma$. The probability distribution for ρ is then $p_\rho(\rho) = 2\rho e^{-\rho^2}$ and the RMS value of ρ is 1.

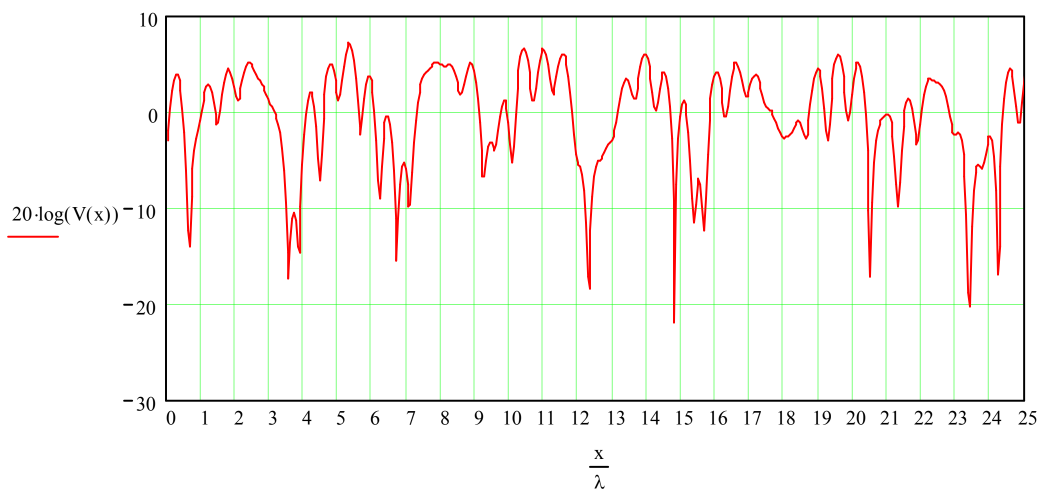

Fig. 2 A Rayleigh fading signal as a function of linear position.

With reference to Fig. 2 an important question is how many times per second the signal will drop below some critical value. The number of times per second that the signal can change is proportional to the sampling rate, hence to f_{max}. The fraction of these that will result in a given value of ρ is proportional to the probability of getting a particular value of ρ, namely, $2\rho e^{-\rho^2}$. A more detailed analysis provides the constant of proportionality, and we have the result that the number of fades per second below the level ρ is

$$N_\rho = \sqrt{2\pi} f_{\max} \rho e^{-\rho^2} \approx \sqrt{2\pi} f_{\max} \rho \tag{5}$$

In the last expression we've used the approximation $e^{-\rho^2} = 1$, which is good to 1%, or better for $\rho \le 0.1$. Usually we are interested in fades corresponding to small values of ρ since these are the cases that cause problems for our link.

Another important issue is how long each of these fades lasts, on average. The fraction of each second that the signal is below a value ρ is given by

$$\int_0^\rho 2 r e^{-r^2} dr = 1 - e^{-\rho^2} \tag{6}$$

Dividing this by the number of fades per second gives us the average time per fade:

$$T_\rho = \frac{e^{\rho^2} - 1}{\sqrt{2\pi} f_{\max} \rho} \tag{7}$$

If $\rho \ll 1$ then $e^{\rho^2} - 1 \approx (1+\rho^2) - 1 = \rho^2$. Since $\rho^2/\rho = \rho$ we have

$$T_\rho \approx \frac{\rho}{\sqrt{2\pi} f_{\max}} \tag{8}$$

This is good to about 1% or better for $\rho \le 0.1$. We note that both N_ρ and T_ρ are proportional to ρ (for small ρ), so deeper fades happen less often and are of shorter duration. N_ρ is proportional to $f_{\max}$ while T_ρ is proportional to $1/f_{\max}$, so fades happen more frequently but are of shorter duration as our velocity increases, or our wavelength decreases.

As an example of when me might be interested in these statistics, if we are operating a digital system with a bit period of T_b and if ρ is the level below which fading will produce a bit error, then T_ρ / T_b is the average number of bits that will be destroyed by a fade. This will occur N_ρ times per second on average. The digital coding scheme employed must be able to operate under these conditions.

3 Diversity Reception

Previously we saw that under Rayleigh fading conditions the probability distribution for received power is

$$p_P(P) = \frac{1}{P_0} e^{-\frac{P}{P_0}} \tag{9}$$

where P_0 is the average received power (all powers are on a linear scale). The probability that the received power will be less than some value $P_{\min}$ is

$$\Pr[P_R < P_{\min}] = \int_0^{P_{\min}} \frac{1}{P_0} e^{-\frac{P}{P_0}} dP = 1 - e^{-\frac{P_{\min}}{P_0}} \tag{10}$$

Imagine that we have two identical antennas far enough apart that they sample independent points of the field. The probability that the received power at both will be less than $P_{\min}$ is just the product of the probabilities that each will receive less than $P_{\min}$, or

$$\Pr[P_{R1}<P_{\min} \text{ and } P_{R2}<P_{\min}]=\left(1-e^{-\frac{P_{\min}}{P_0}}\right)^2 \tag{11}$$

Likewise, if we have *N* identical antennas sampling *N* independent field points we'd have this same factor to the *N* power. The idea of switched diversity is to continuously monitor the signal strength at multiple antennas and choose the one with the largest signal. The only way the result can be less than $P_{\min}$ is if all *N* antennas are receiving less than Pmin. For a switched diversity system we therefore have

$$\Pr[P_R<P_{\min}]=\left(1-e^{-\frac{P_{\min}}{P_0}}\right)^N \tag{12}$$

If we assume that $P_{\min}/P_0 \ll 1$ then using $e^{-x}\approx 1-x$ we can write

$$\Pr[P_R<P_{\min}]=\left(\frac{P_{\min}}{P_0}\right)^N \tag{13}$$

Consider a single antenna with an average received power of P_0 , and a switched diversity system with *N* antennas and an average received power reduced by a factor of $1/G_{\text{div}}$, that is P_0/G_{div}. Equating the probabilities of fades below $P_{\min}$ we find

$$\frac{P_{\min}}{P_0}=\left(\frac{P_{\min}}{P_0/G_{\text{div}}}\right)^N \tag{14}$$

from which

$$G_{\text{div}}=\left(\frac{P_{\min}}{P_0}\right)^{\frac{N-1}{N}} \tag{15}$$

In other words, with respect to this criterion, a switched diversity system can tolerate a reduction in power of $1/G_{\text{div}}$ while maintaining the same performance. The effect is the same as if there were a single antenna but the power level was increased by the factor G_{div} – the "diversity gain." This is a critical benefit.

Consider the following situation. A basestation and mobile each use single antennas for both transmit and receive. By reciprocity, therefore,

$$\begin{aligned} P_{RM}&=P_{TB}-L \\ P_{RB}&=P_{TM}-L \end{aligned} \tag{16}$$

where *L* is the system loss. Typically P_{TM} is limited by battery power considerations. On the other hand P_{TB} is not. In principle we can pump up the base transmitter power as much as we want to get a good received signal at the mobile, but we can't easily pump up the mobile transmitter power. Therefore the uplink would tend to impose the primary limit on the distance *r* at which we could communicate. However, if we employ diversity reception at the base station then we can achieve the same result as increasing the mobile power by the factor G_{div}. Typical cellular systems use $N=2$ which accounts for the three antennas you often see at a basestation – one for transmission and two for diversity reception.

On a logarithmic scale we have

$$G_{\text{div,dB}}=\frac{N-1}{N}\left(P_{0,\,\text{dBm}}-P_{\text{min,dBm}}\right) \tag{17}$$

i.e., for $N=2$ the diversity gain is half of the number of dB's by which the average received signal exceeds the threshold.

Example 1: A basestation transmits 5 W or 37 dBm of power. A receiver needs $P_{\text{min}}=-110\,\text{dBm}$ of power to operate properly. With Rayleigh fading an average received signal of $P_0=-90\,\text{dBm}$ results in fades below P_{min} a fraction of the time of $1-e^{-0.01}\approx 0.01$, i.e., 1% of the time. If a mobile is receiving a barely acceptable signal from the basestation, how much power does it need to transmit to produce a barely acceptable signal at the basestation if the basestation uses $N=2$ diversity reception?

If no diversity reception were used, then of course the mobile would also have to transmit 37 dBm to produce –90 dBm at the basestation. However, $P_0/P_{\text{min}}=100$ and from (17) with $N=2$ we get $G_{\text{div}}=10\,\text{dB}$. So we can tolerate an average received signal of $-90\,\text{dBm}-10\,\text{dB}=-100\,\text{dBm}$ and still keep our 1% probability of fades below P_{min}. Consequently the mobile can transmit 10 dB less than the basestation, or 27 dBm, which is 0.5 W.

4 References

1. Rappaport, T. S., *Wireless Communications: Principles and Practice*, Prentice Hall, 1996, ISBN 0-13-375536-3.
2. Saunders, S. R., *Antennas and Propagation for Wireless Communication Systems*, Wiley, 1999, ISBN 0-471-98609-7.