Lecture 8

Small Scale Fading

1 Introduction

In a previous lesson we considered the multipath phenomenon. We saw that in a nontrivial environment we should expect there to be several reflected field components in addition to the direct-path component we'd get in free space. These reflected waves will typically be traveling in different directions, so our received field will look like the sum of a number of randomly oriented plane waves. The result is that at any point in space the field amplitude will be a random phasor sum. In this lesson we want to consider the resulting field amplitude probability distribution.

Field amplitude is related to receiver voltage by the antenna factor (AF). This is defined as

$$AF=\frac{|E|}{|V|} \tag{1}$$

where $|E|$ is the magnitude of the electric field at the receiving antenna (units of volts-per-meter) and $|V|$ is the magnitude of the voltage in the receiver (units of volts). The *AF* therefore has units of m^{-1} and is related to antenna gain as follows. The intensity of a field is $|E|^2/(2\eta_0)$ with $\eta_0=377\,\Omega$ the impedance of free space. Received power is $|V|^2/(2Z_0)$ with $Z_0=50\,\Omega$ the receiver impedance. Received power is also field intensity times antenna area. Effective area is related to gain by $A=G\lambda^2/(4\pi)$. Therefore

$$\frac{|V|^2}{2Z_0}=\frac{|E|}{2\eta_0}\frac{G\lambda^2}{4\pi} \tag{2}$$

from which

$$AF=\frac{|E|}{|V|}=\frac{\sqrt{4\pi\eta_0/Z_0}}{\lambda\sqrt{G}}=\frac{9.73}{\lambda\sqrt{G}} \tag{3}$$

We see that AF is simply another way to represent antenna gain. Its usefulness is that it allows us to directly convert receiver voltage into field amplitude.

2 Rayleigh Fading

Rayleigh fading is the limit of the interference of an infinitely large number of randomly oriented field components. Before we delve into the theory let's look at simulation and experiment to see what we're talking about. Our simulation will generate a relatively large number of plane waves with random directions of propagation and relative phases. The following figure shows the Mathcad code to generate the simulation. We'll discuss the theoretical "Rayleigh" distribution below.

2.1 Simulation

N plane waves of equal amplitude $N := 20$ $\quad i := 0..N-1$

random directions and phases $\theta_i := \mathrm{rnd}(2\cdot\pi)$ $\quad \phi_i := \mathrm{rnd}(2\cdot\pi)$

field amplitude $f(x,y) := \sum_i e^{j\cdot\phi_i}\cdot e^{-j\cdot 2\cdot\pi\cdot(x\cos(\theta_i)+y\cdot\sin(\theta_i))}$

sample field for plotting $M := 50$ $\quad n := 0..M$ $\quad m := 0..M$ $\quad a_{n,m} := \left|f\left(\frac{n}{10},\frac{m}{10}\right)\right|$

histogram of amplitude values $v_{n\cdot M+m} := a_{n,m}$ $\quad N := 100$ $\quad i := 0..N$ $\quad \mathrm{int}_i := \frac{i}{10}$

theoretical distribution $\mathrm{Rayleigh}(r,\sigma) := \frac{r}{\sigma^2}\cdot e^{-\frac{r^2}{2\cdot\sigma^2}}$ $\quad \sigma := \frac{\mathrm{mean}(v)}{1.2533}$ $\quad \sigma = 3.401$

Fig. 1 Mathcad simulation of Rayleigh fading.

The next figure shows the resulting simulated field amplitude as a function of spatial location and a histogram of the field amplitude values. We can see that the field amplitude varies considerably over a square that is only one wavelength on a side. For example, at 1900 MHz one wavelength is only about 16 cm – about the size of a soccer ball.

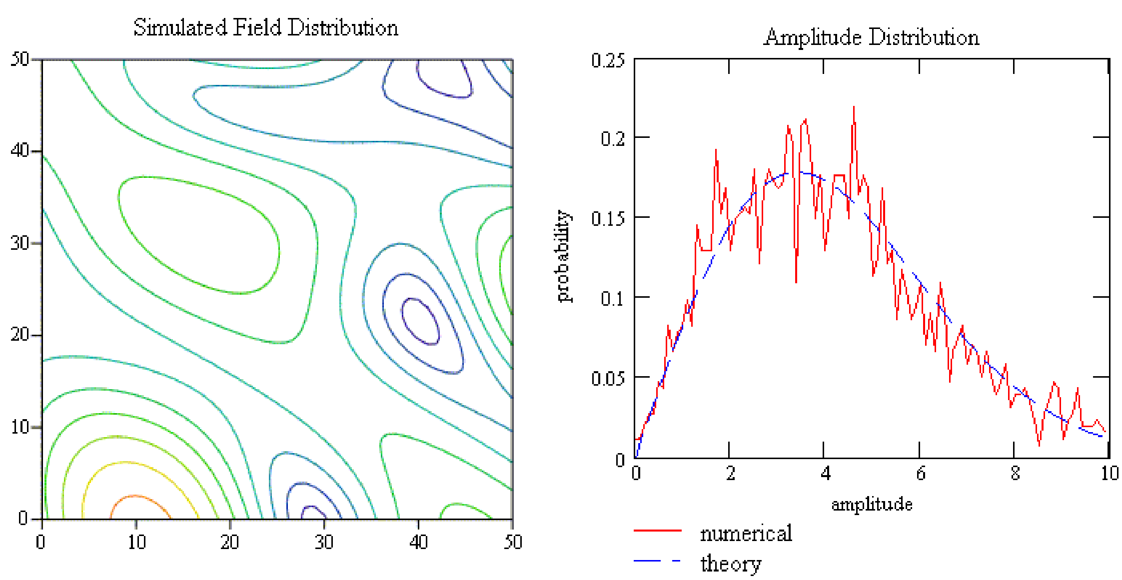

Fig. 2 Results of above simulation. The contour plot of field distribution is one wavelength on a side. The numerical amplitude distribution is simply a histogram of the field distribution. The "theory" curve is a Rayleigh distribution.

The amplitude distribution has the appearance of a skewed Bell Curve. As we'll see below, this is called a "Rayleigh Distribution."

2.2 Experiment

The following figure displays the results measuring AMPS control channel field strengths over 8-inch square grids and producing contour plots and histograms from these measurements. The contour plots are reminiscent of that in Fig. 8.2. The field amplitude histograms are not precisely Rayleigh distributed but qualitatively similar.

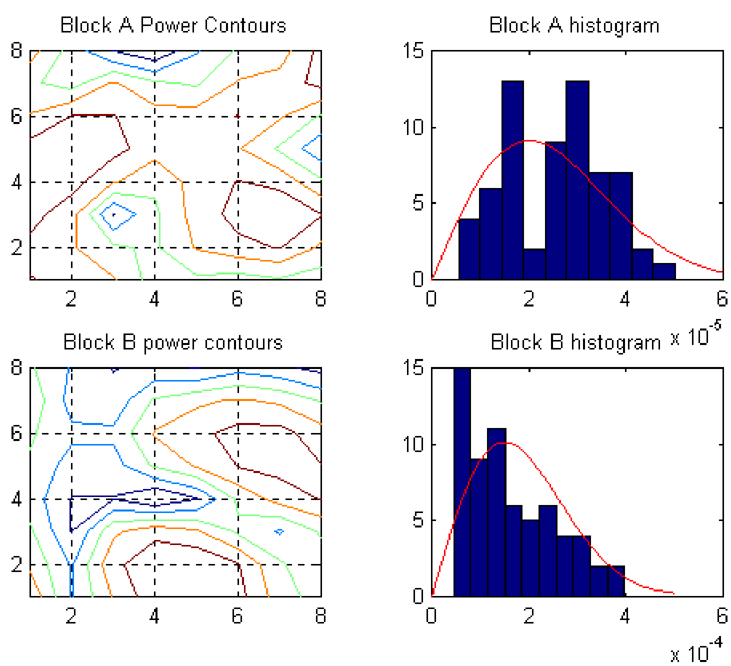

Fig. 3 Field strength measurements in EME206 of A and B block AMPS control channels. Measurements were made over 8 by 8 inch grids corresponding to approximately 0.6 wavelengths.

3 Theory

Our model for Rayleigh fading is an infinite number of randomly oriented plane waves interfering. If we plop down at some arbitrary point in space, each of these plane waves will have some random phase. For simplicity let's assume each plane wave has the same amplitude α. (A more involved derivation with random amplitudes leads to the same result we'll get here.) Then for either the field amplitude or the receiver voltage, we have a random phasor sum of the form

$$a\,e^{j\theta}=\sum_{k=1}^{N}\alpha\,e^{j\phi_k} \tag{4}$$

Call the real and imaginary parts of this *r* and *i* we have

$$r=\alpha\sum_{k=1}^{N}\cos\phi_k$$
$$i=\alpha\sum_{k=1}^{N}\sin\phi_k \tag{5}$$

By the Central Limit Theorem, both r and i should tend to Gaussian random variables for large N. Since the average value of both sin and cos are zero, the mean values of r and i are zero. The variance of r can be calculated as (angled brackets denote mean value)

$$\begin{aligned}\sigma^2&=\alpha^2\left\langle\sum_{k=1}^{N}\cos\phi_k\sum_{m=1}^{N}\cos\phi_m\right\rangle\\&=\alpha^2\sum_{k=1}^{N}\cos^2\phi_k\\&=\frac{N}{2}\alpha^2\end{aligned} \tag{6}$$

because $\langle\cos\phi_k\cos\phi_m\rangle=0$ when $k\neq m$. The variance of i is the same. Also $\langle r\,i\rangle=0$ because $\langle\cos\phi_k\sin\phi_m\rangle=0$ for any k and m. A theorem from statistics tells us that under these conditions r and i are independent Gaussian random variables. Their joint probability distribution is

$$\begin{aligned}p_{ri}(r,i)&=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{i}{\sigma}\right)^2}\\&=\frac{1}{2\pi\sigma^2}e^{-\frac{1}{2}\frac{r^2+i^2}{\sigma^2}}\end{aligned} \tag{7}$$

If r and i are the real and imaginary parts of the received voltage phasor we can write $r=V\cos\theta$ and $i=V\sin\theta$ with V the voltage amplitude. We can use this change of variables to write

$$\frac{1}{2\pi\sigma^2}e^{-\frac{1}{2}\frac{r^2+i^2}{\sigma^2}}dr\,di=\frac{V}{\sigma^2}e^{-\frac{V^2}{2\sigma^2}}dV \tag{8}$$

where we've used $dr\,di=V\,dV\,d\theta$. This gives us the probability distribution for the received voltage:

$$p_V(V)=\frac{V}{\sigma^2}e^{-\frac{V^2}{2\sigma^2}} \tag{9}$$

We call this the Rayleigh distribution. It gives the probability of obtaining a given voltage in a receiver in an environment consisting of a large number of randomly oriented field components. The expected value of V is $\sigma\sqrt{\pi/2}$.

If we are interested in received power we can use $P\propto V^2=r^2+i^2$ to get $P_0=\langle P\rangle\propto 2\sigma^2$ and $dP\propto 2\,V\,dV$. Therefore

$$\frac{V}{\sigma^2}e^{-\frac{V^2}{2\sigma^2}}dV=\frac{1}{P_0}e^{-\frac{P}{P_0}}dP \tag{10}$$

and the probability distribution for power is

$$p_P(P)=\frac{1}{P_0}e^{-\frac{P}{P_0}}dP \tag{11}$$

The expected value of *P* is P_0. However, notice that the most likely received power is zero!

Sometimes you need to generate random numbers with a Rayleigh distribution. One way to do this is to take a random variable *x* that is uniformly distributed over $[0,1]$ and pass it through the function $y=\sigma\sqrt{2\ln\frac{1}{x}}$. The result is that *y* will be Rayleigh distributed.

4 Other Types of Fading

Rayleigh fading is the limiting case of a large number of interfering plane waves. The other extreme would be the case where the field is a single plane wave, such as would occur in free space propagation. (The real field would be more like a spherical wave, but over a small region of space far from the transmitter it would look essential like a plane wave.) In this case there would be no fading and the received signal would be perfectly constant.

In between these extremes is the case where the field consists of a single strong plane wave with many weaker random waves. We could treat this case by using $r=r_0+\alpha\sum_{k=1}^{N}\cos\phi_k$ in (5) where r_0 is the amplitude of the single strong wave. The result is referred to as *Ricean fading*.

Since Rayleigh fading represents the worst case fading scenario, it is the model most often used when designing wireless systems. If a system will work under Rayleigh fading then it will almost certainly work under less severe conditions. For this reason we will only consider Rayleigh fading in this course.

5 References

1. Goodman, J. W., *Statistical Optics*. Wiley 1985. ISBN 0-471-01502-4. See section 2.9 “Random phasor sums.”
2. Press, W. H., B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes in C*, Cambridge 1988, ISBN 0-521-35465-X. See section 7.1 “Transform Method.”