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Quantum Computing for Computer Scientists

The gate quantum computation model

Why learn quantum computing?

- Quantum supremacy expected this year
  - Microsoft, Google, Intel, IBM all investing in quantum computer development
- Several exciting applications already known
  - Efficiently factor large composite numbers, breaking RSA encryption (Shor's algorithm, 1994)
  - Search an unordered list in $O(\sqrt{n})$ time (Grover's algorithm, 1996)
  - Believed exponential speedup in simulating quantum mechanical systems
- Intellectually interesting – quantum mechanics is outside your intuition!
  - Get a small glimpse of what you don't know you don't know

Learning objectives

- Representing computation with basic linear algebra (vectors and matrices)
- Qbits, superposition, and quantum logic gates
- The simplest problem where a quantum computer beats a classical computer
- Bonus topics: quantum entanglement and teleportation

Representing classical bits as a vector

One bit with the value 0, also written as $|0\rangle$ (Dirac vector notation)

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

One bit with the value 1, also written as $|1\rangle$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Review: matrix multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

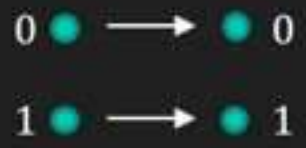
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Operations on one classical bit (cbit)

Identity	$f(x) = x$		$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$
Negation	$f(x) = \neg x$		$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$
Constant-0	$f(x) = 0$		$\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$
Constant-1	$f(x) = 1$		$\begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$

Reversible computing

- Reversible means given the operation and output value, you can find the input value
  - For $Ax = b$, given b and A, you can uniquely find x
- Operations which permute are reversible; operations which erase & overwrite are not
  - Identity and Negation are reversible
  - Constant-0 and Constant-1 are not reversible
- Quantum computers use only reversible operations, so we will only care about those
  - In fact, all quantum operators *are their own inverses*

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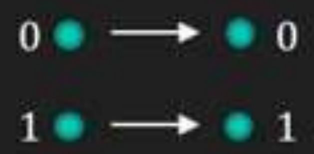
Identity	$f(x) = x$		$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$
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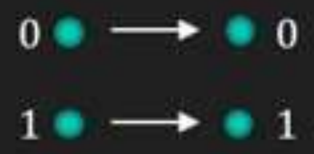
Identity	$f(x) = x$		$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$
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Review: tensor product of vectors

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \\ x_1 \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} x_0 y_0 \\ x_0 y_1 \\ x_1 y_0 \\ x_1 y_1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \otimes \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_0 y_0 z_0 \\ x_0 y_0 z_1 \\ x_0 y_1 z_0 \\ x_0 y_1 z_1 \\ x_1 y_0 z_0 \\ x_1 y_0 z_1 \\ x_1 y_1 z_0 \\ x_1 y_1 z_1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Representing multiple cbits

$$|00\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |01\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

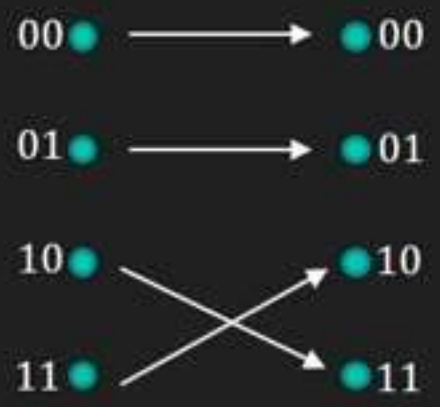
$$|10\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |11\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$|4\rangle = |100\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix}$$

- We call this tensored representation the **product state**
- We can **factor** the product state back into the **individual state** representation
- The product state of n bits is a vector of size 2^n

Operations on multiple cbits: CNOT

- Operates on pairs of bits, one of which is the "control" bit and the other the "target" bit
- If the control bit is 1, then the target bit is flipped
- If the control bit is 0, then the target bit is unchanged
- The control bit is always unchanged
- With most-significant bit as control and least-significant bit as target, action is as follows:



$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Operations on multiple cbits: CNOT

$$C|10\rangle = C\left(\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = |11\rangle$$

$$C|11\rangle = C\left(\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = |10\rangle$$

Operations on multiple cbits: CNOT

$$C|00\rangle = C\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = |00\rangle$$

$$C|01\rangle = C\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = |01\rangle$$

Recap

- We represent classical bits in vector form as $\binom{1}{0}$ for 0 and $\binom{0}{1}$ for 1
- Operations on bits are represented by matrix multiplication on bit vectors
- Quantum computers only use reversible operations
- Multi-bit states are written as the tensor product of single-bit vectors
- The CNOT gate is a fundamental building block of reversible computing

Qbits and superposition

- Surprise! We've actually been using qbits all along!
- The cbit vectors we've been using are just special cases of qbit vectors
- A qbit is represented by $\binom{a}{b}$ where a and b are Complex numbers and $\|a\|^2 + \|b\|^2 = 1$
  - The cbit vectors $\binom{1}{0}$ and $\binom{0}{1}$ fit within this definition
  - Don't worry! For this presentation, we'll only use familiar Real numbers.
- Example qbit values:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Qbits and superposition

- How can a qbit to have a value which is not 0 or 1? This is called superposition.
- Superposition means the qbit is both 0 and 1 and the same time
- When we **measure** the qbit, it **collapses** to an actual value of 0 or 1
  - We usually do this at the end of a quantum computation to get the result
- If a qbit has value $\binom{a}{b}$ then it collapses to 0 with probability $||a||^2$ and 1 with probability $||b||^2$
  - For example, qbit $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ has a $\left\|\frac{1}{\sqrt{2}}\right\|^2 = \frac{1}{2}$ chance of collapsing to 0 or 1 (coin flip)
  - The qbit $\binom{1}{0}$ has a 100% chance of collapsing to 0, and $\binom{0}{1}$ has a 100% chance of collapsing to 1

Qbits and superposition

- Multiple qbits are similarly represented by the tensor product $\binom{a}{b} \otimes \binom{c}{d} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$
  - Note that $\|ac\|^2 + \|ad\|^2 + \|bc\|^2 + \|bd\|^2 = 1$
- For example, the system $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ (note that $\left\|\frac{1}{2}\right\|^2 = \frac{1}{4}$, and $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$)
  - There's a ¼ chance each of collapsing to $|00\rangle, |01\rangle, |10\rangle$, or $|11\rangle$

Operations on qbits

- How do we operate on qbits? The same way we operate on cbits: with matrices!
- All the matrix operators we've seen also work on qbits (bit flip, CNOT, etc.)
- Matrix operators model the effect of some device which manipulates qbit spin/polarization without measuring and collapsing it

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$$

- There are several important matrix operators which only make sense in a quantum context

Qbits and superposition

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- There are several important matrix operators which only make sense in a quantum context

The Hadamard gate

- The Hadamard gate takes a 0- or 1-bit and puts it into exactly equal superposition

$$H|0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$H|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

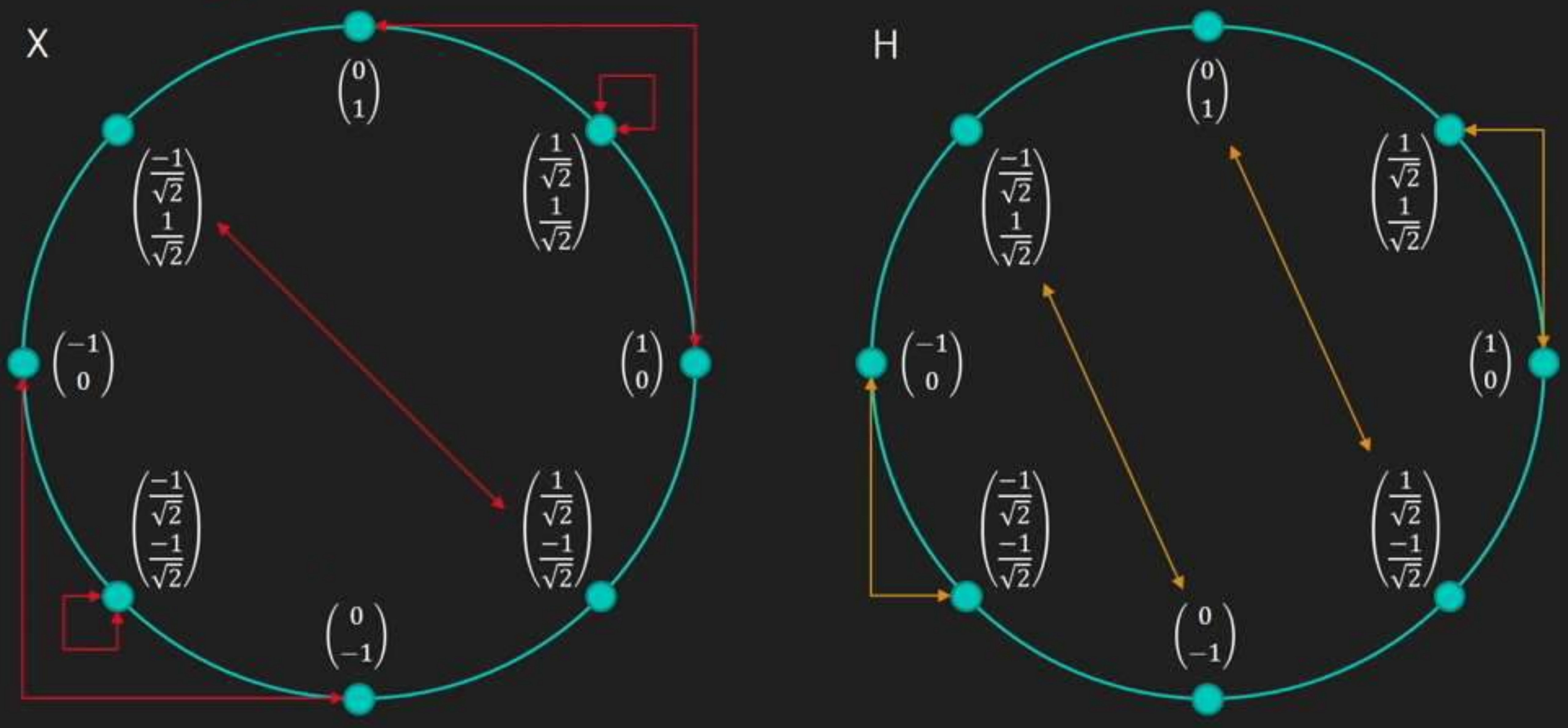
The Hadamard gate

- The Hadamard gate also takes a qbit in exactly-equal superposition, and transforms it into a 0- or 1-bit! (This should be unsurprising – remember operations are their own inverse!)

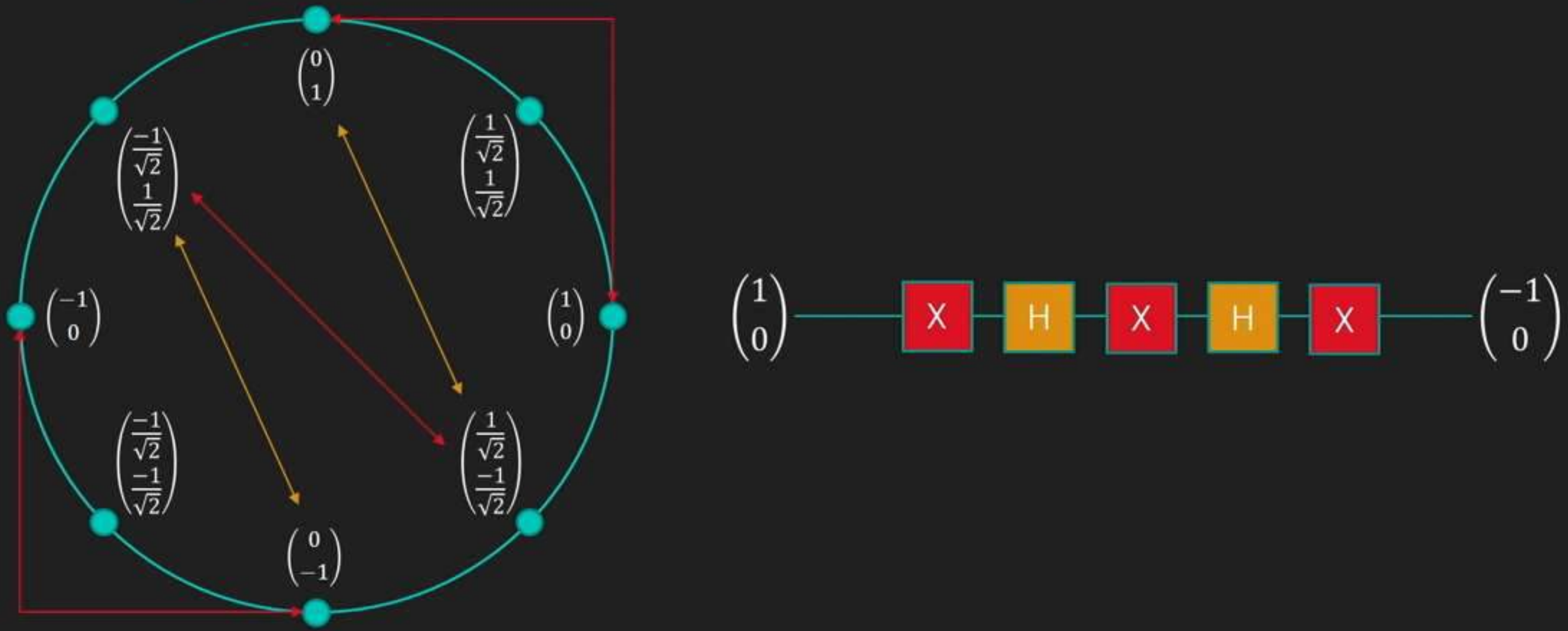
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- We can transition out of superposition without measurement!
- We can thus structure quantum computation deterministically instead of probabilistically

The unit circle state machine



The unit circle state machine



Recap

- Cbits are just a special case of qbits, which are 2-vectors of Complex numbers
- Qbits can be in superposition, and are probabilistically collapsed to cbits by measurement
- Multi-qbit systems are tensor products of single-qbit systems, like with cbits
- Matrices represent operations on qbits, same as with cbits
- The Hadamard gate takes 0- and 1-bits to equal superposition, and back
- We can think of qbits and their operations as forming a state machine on the unit circle
  - Actually the unit sphere if we use complex numbers

The Deutsch oracle

- Imagine someone gives you a black box containing a function on one bit
  - Recall! What are the four possible functions on one bit?
- You don't know which function is inside the box, but can try inputs and see outputs
- How many queries would it take to determine the function on a classical computer?
- How many on a quantum computer?

The Deutsch oracle

- What if you want to check whether the unknown function is constant, or variable?
  - Constant-0 & constant-1 are constant, identity & negation are variable
- How many queries would it take on a classical computer?
- How many on a quantum computer?

The Deutsch oracle

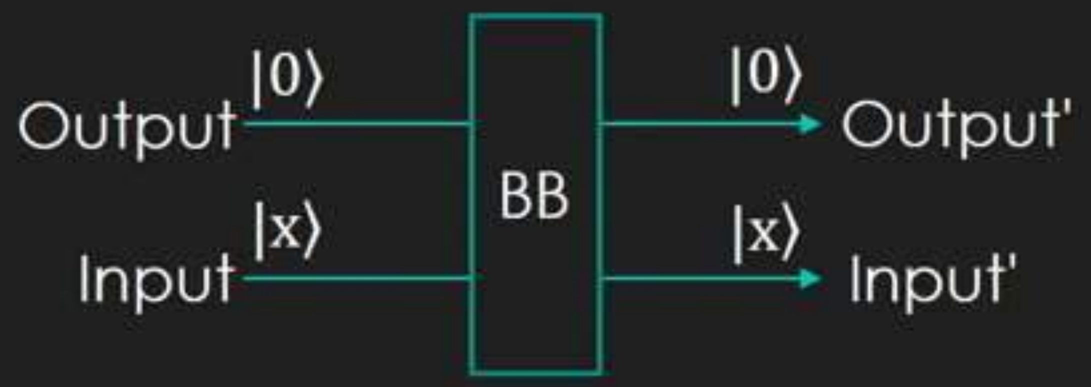
- How can it be done in a single query!?
- We can do it with the magic of superposition!
- First, we have to define what each of the four functions look like on a quantum computer
  - We have an immediate problem with the constant functions

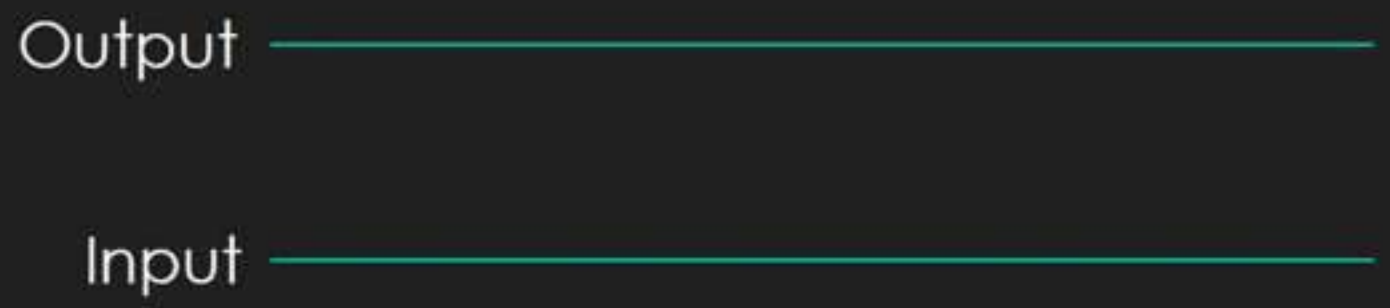
The Deutsch oracle

- How do we write nonreversible functions in a reversible way?
- Common hack: add an additional **output qbit** to which the function action is applied
- We thus have to rewire our black box:

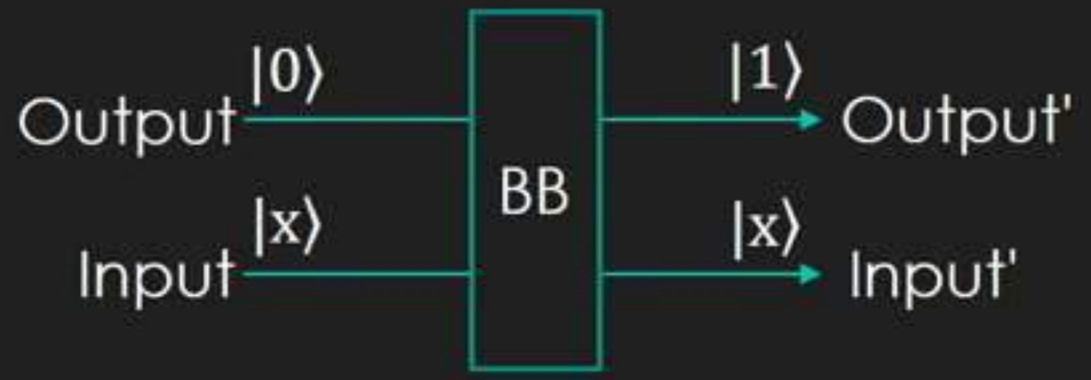
- The black box leaves the **input qbit** unchanged, writing function output to **output qbit**

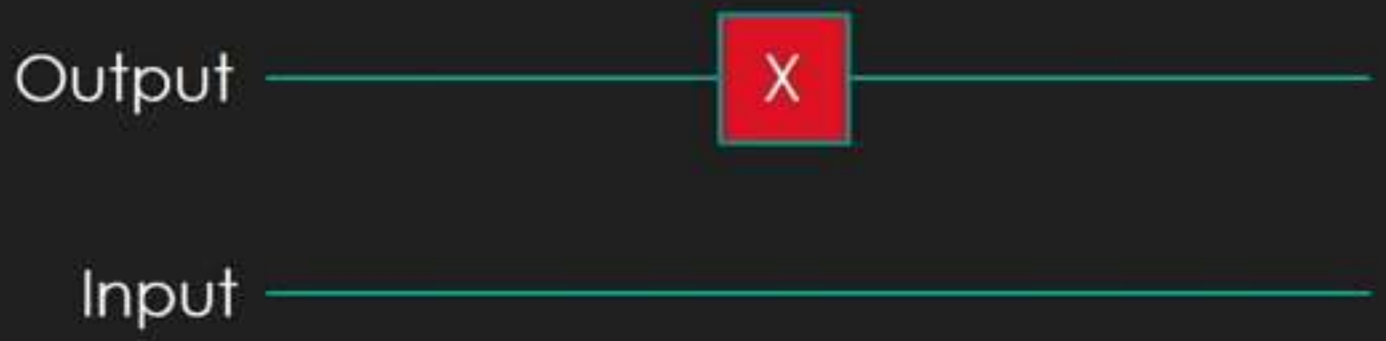
The Deutsch oracle: constant-0





The Deutsch oracle: constant-1





The Deutsch oracle

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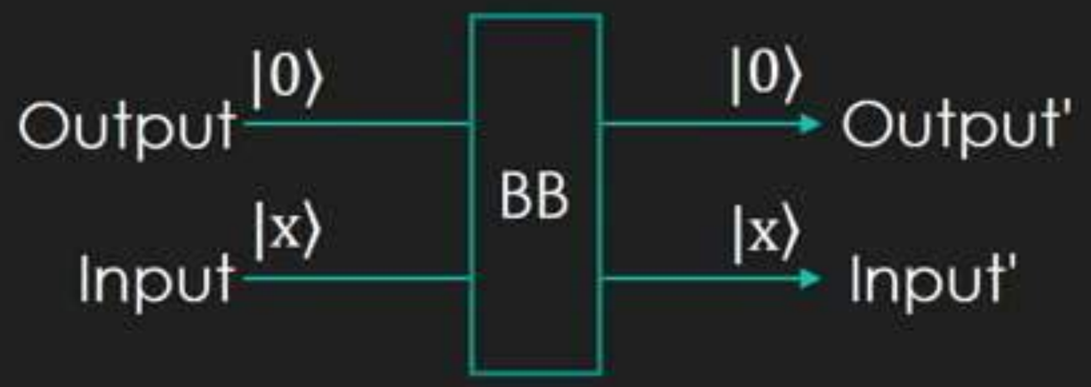
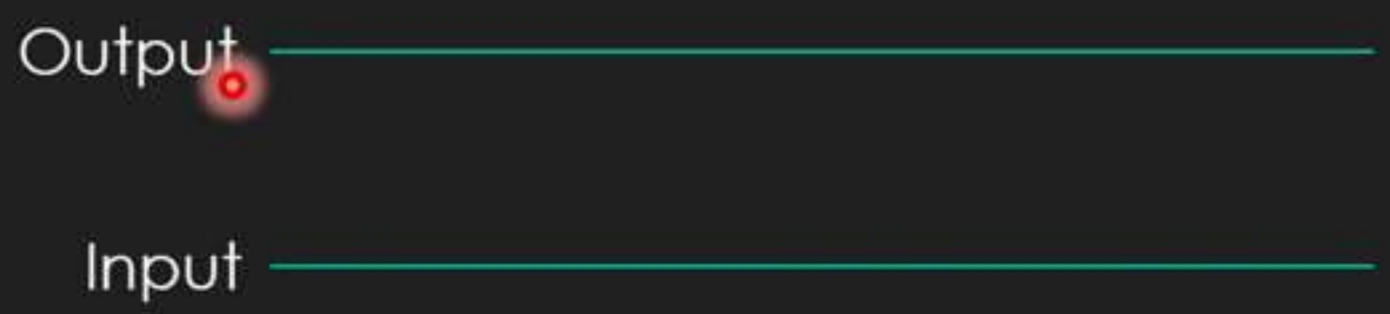
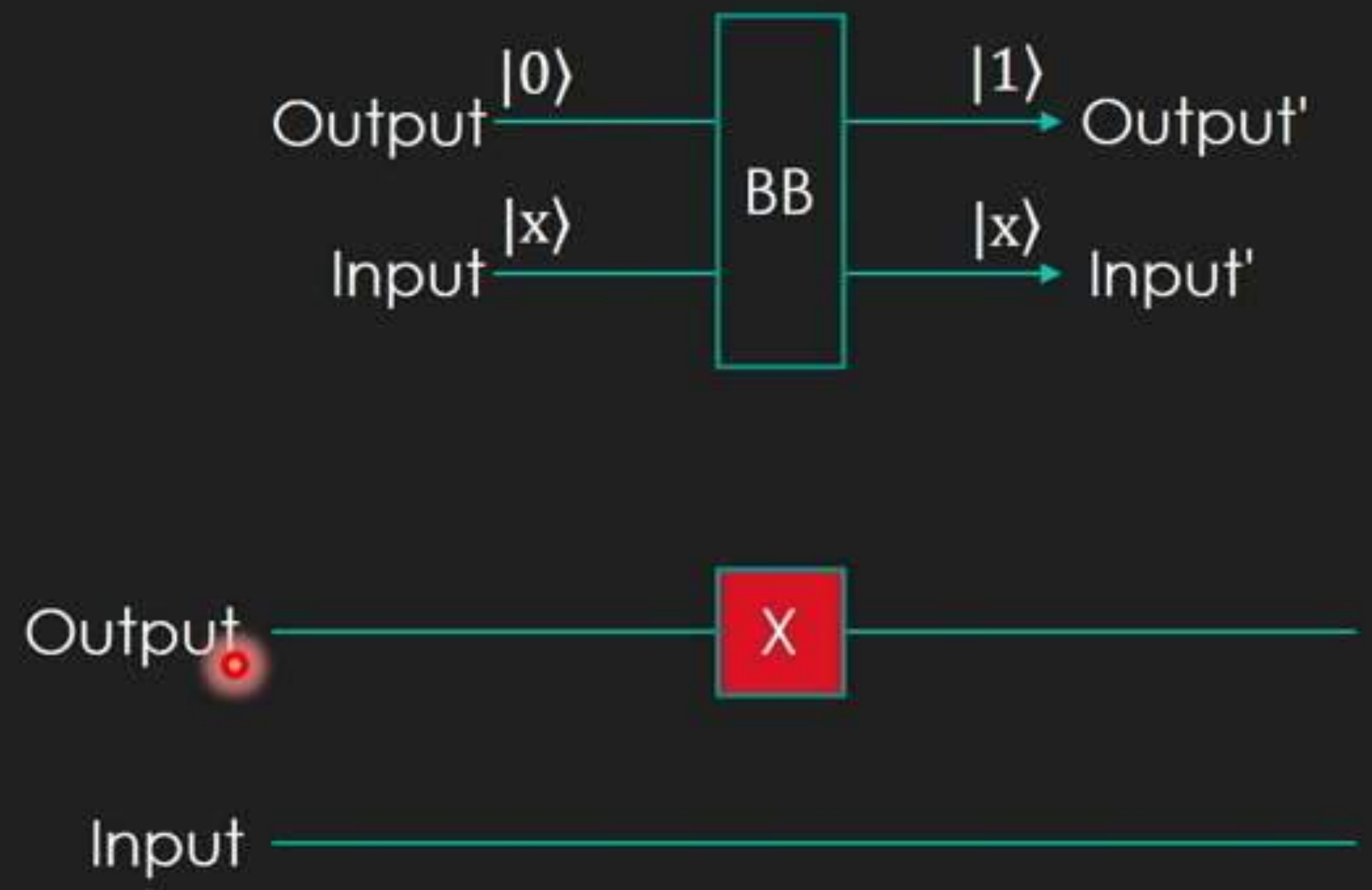
The Deutsch oracle: constant-0

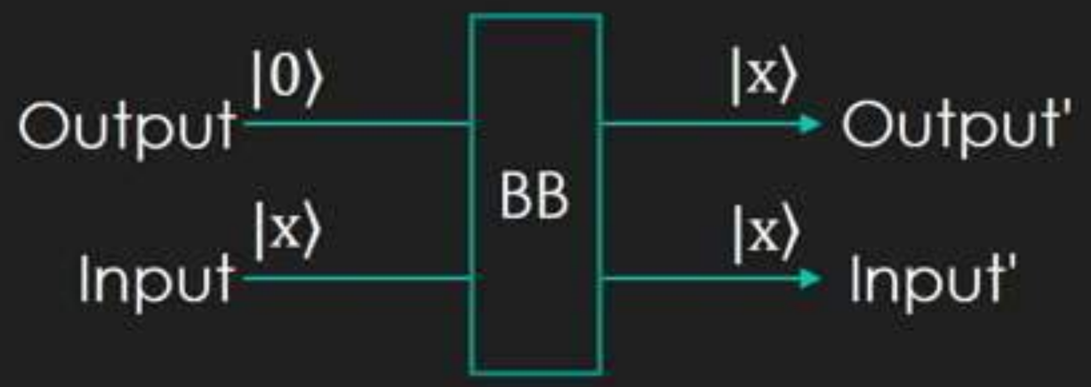




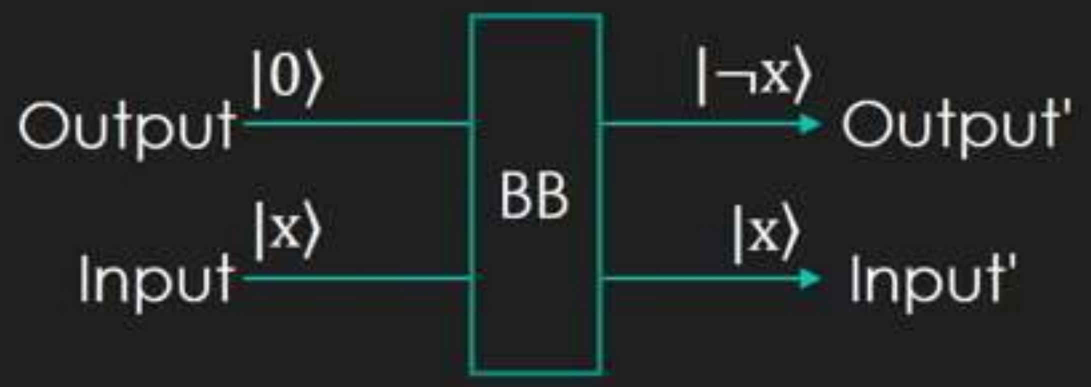
The Deutsch oracle: constant-1



The Deutsch oracle: identity

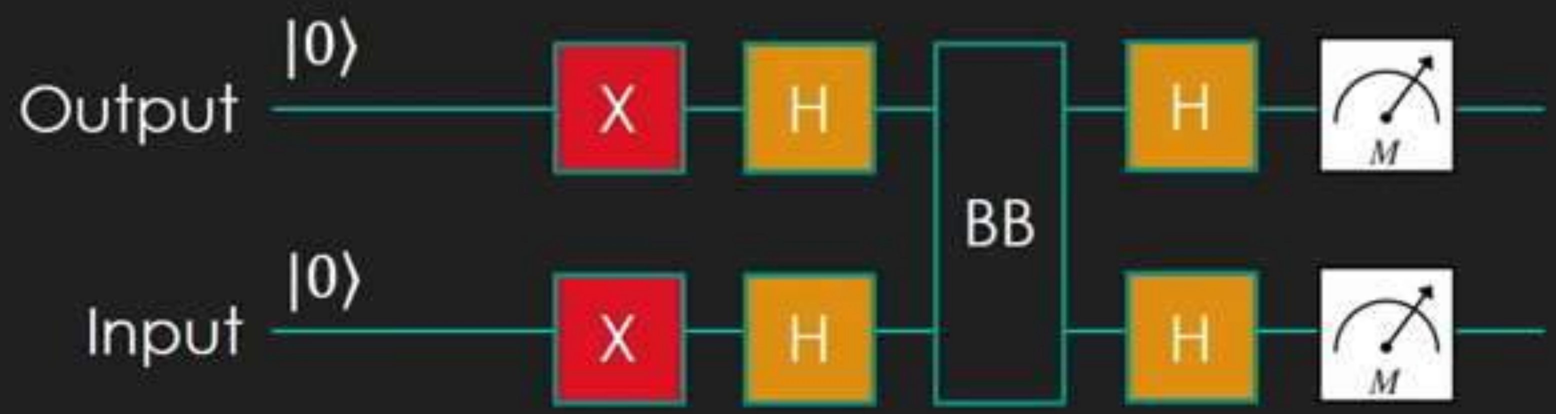


The Deutsch oracle: negation



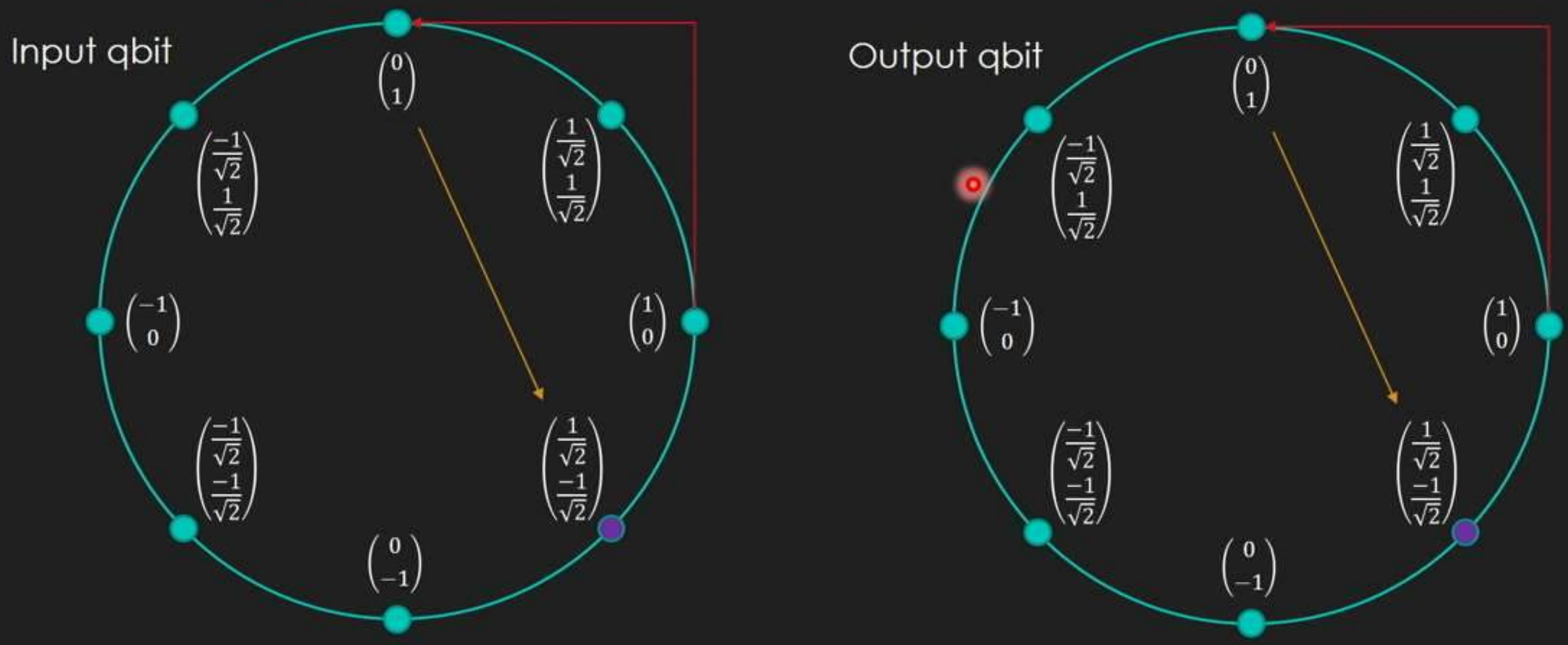
The Deutsch oracle

- How do we solve it on a quantum computer in one query?

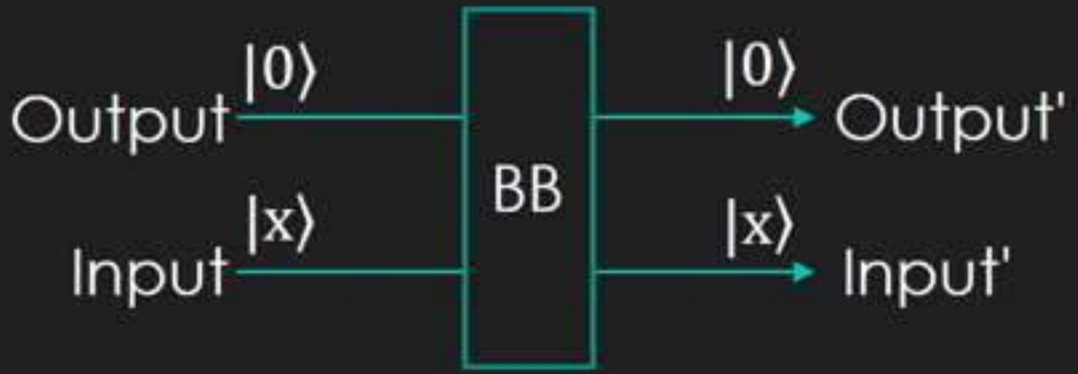


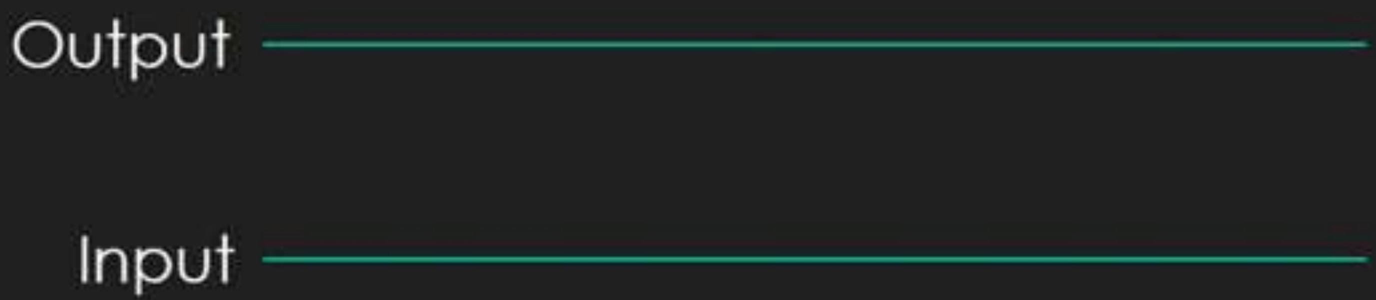
- If the black-box function is constant, system will be in state $|11\rangle$ after measurement
- If the black-box function is variable, system will be in state $|01\rangle$ after measurement

The Deutsch oracle: preprocessing

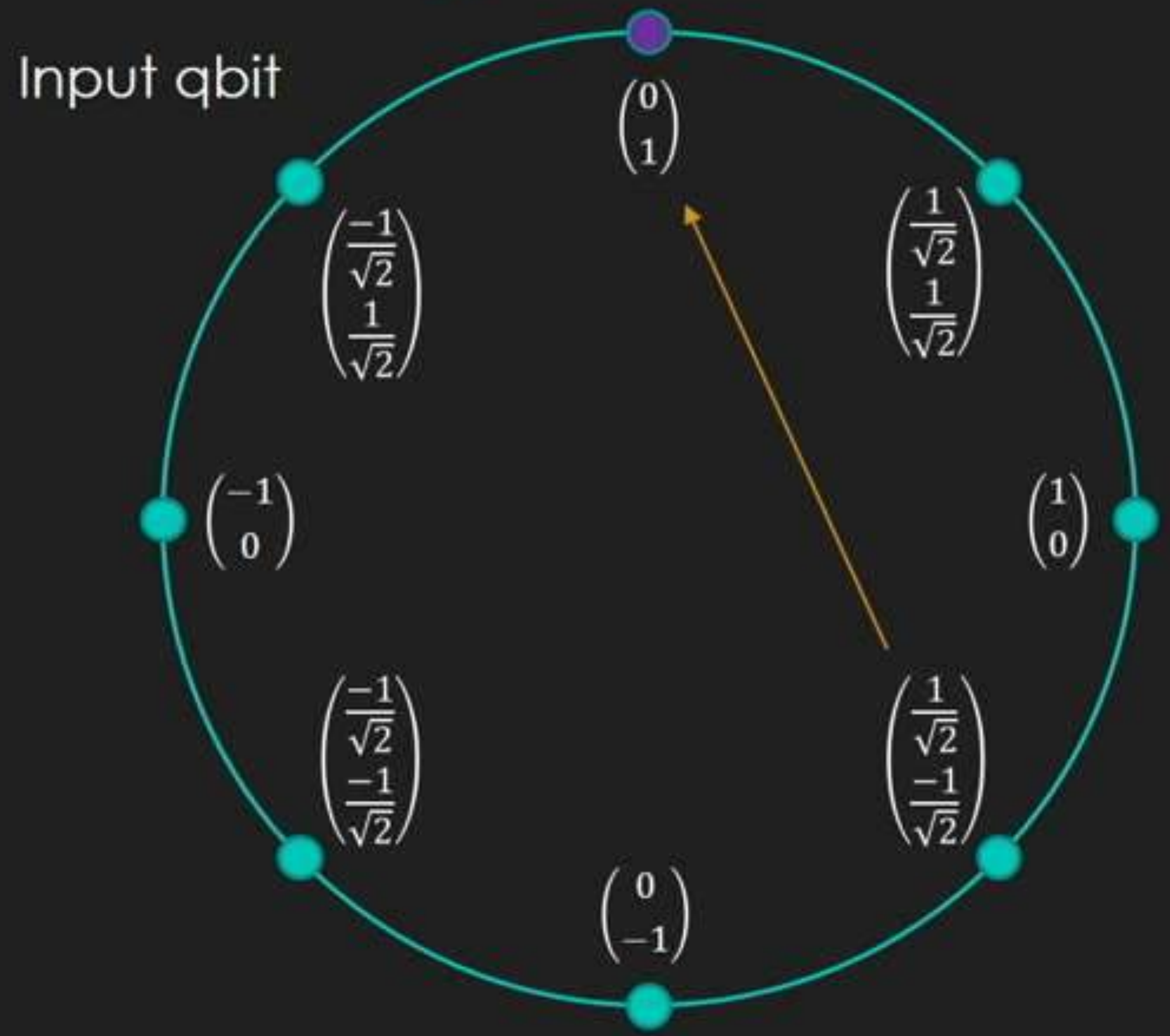


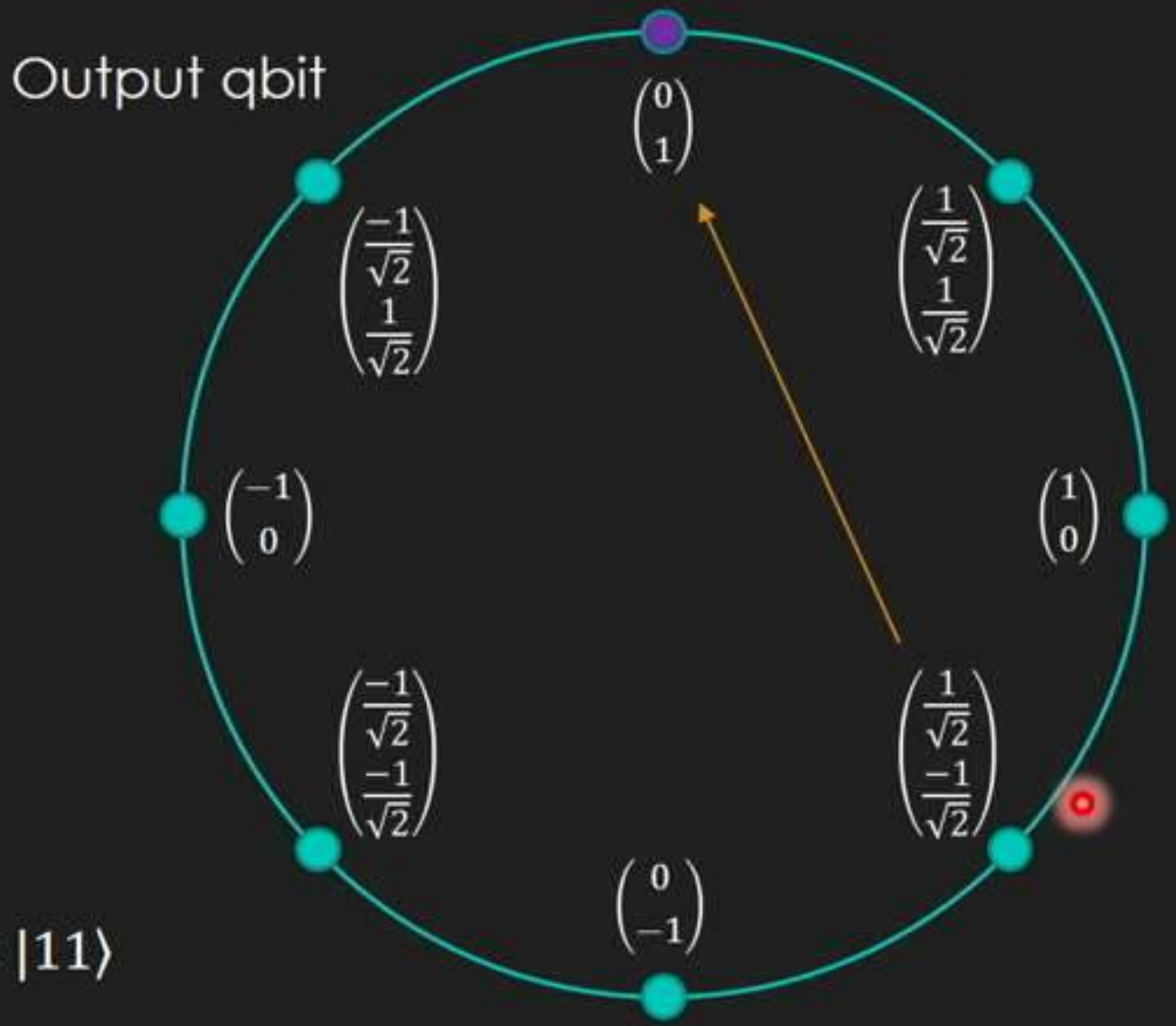
The Deutsch oracle: constant-0





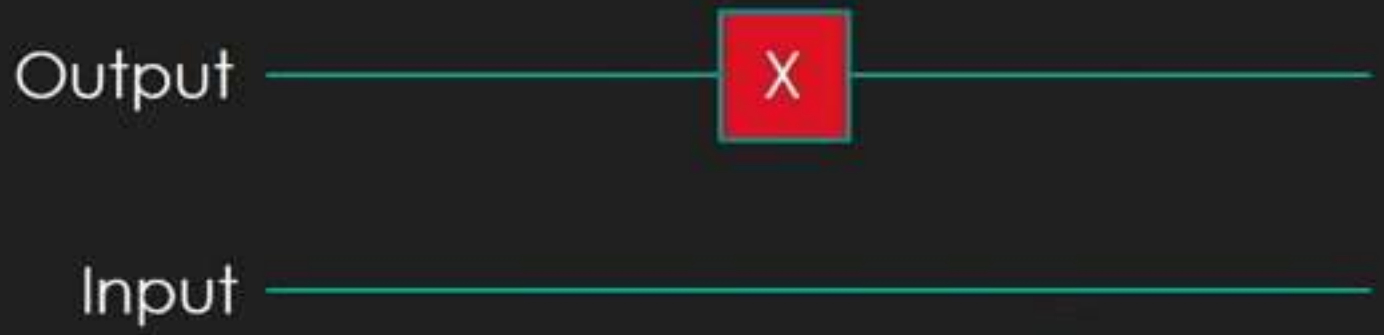
The Deutsch oracle: constant-0



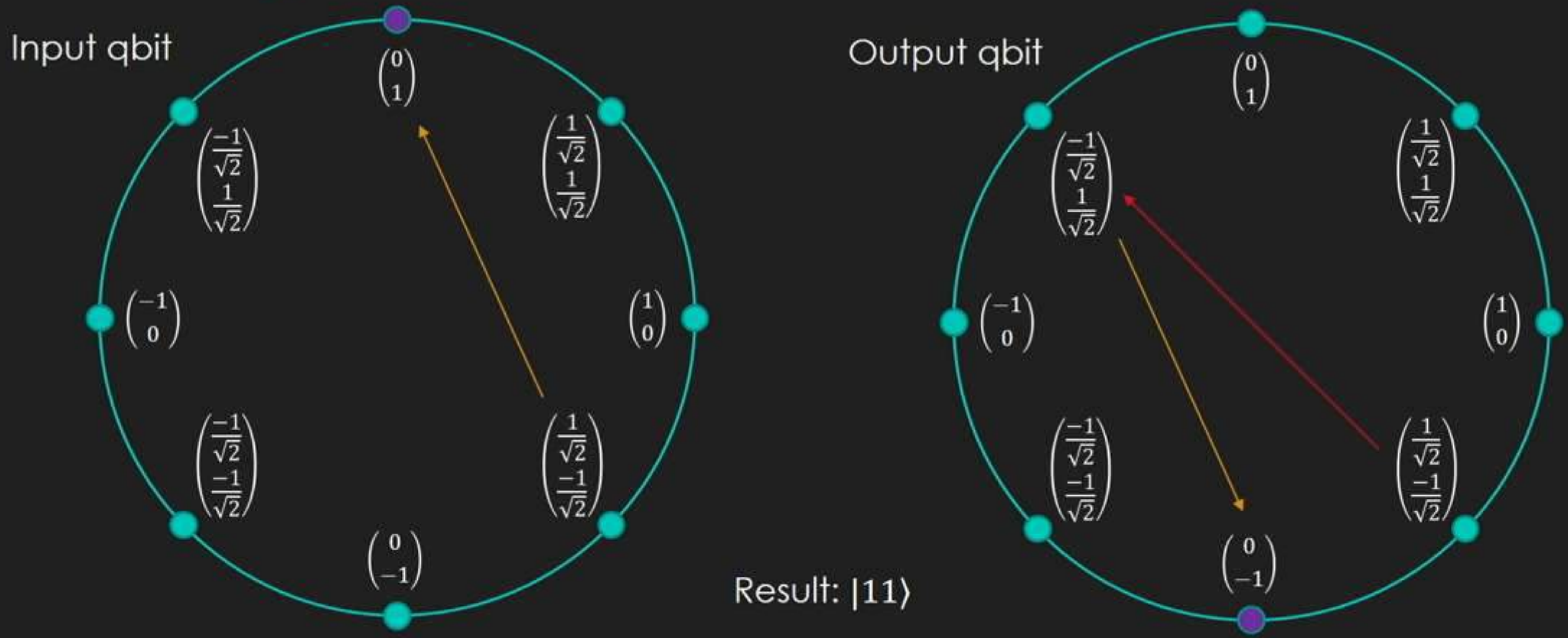


Result: $|11\rangle$

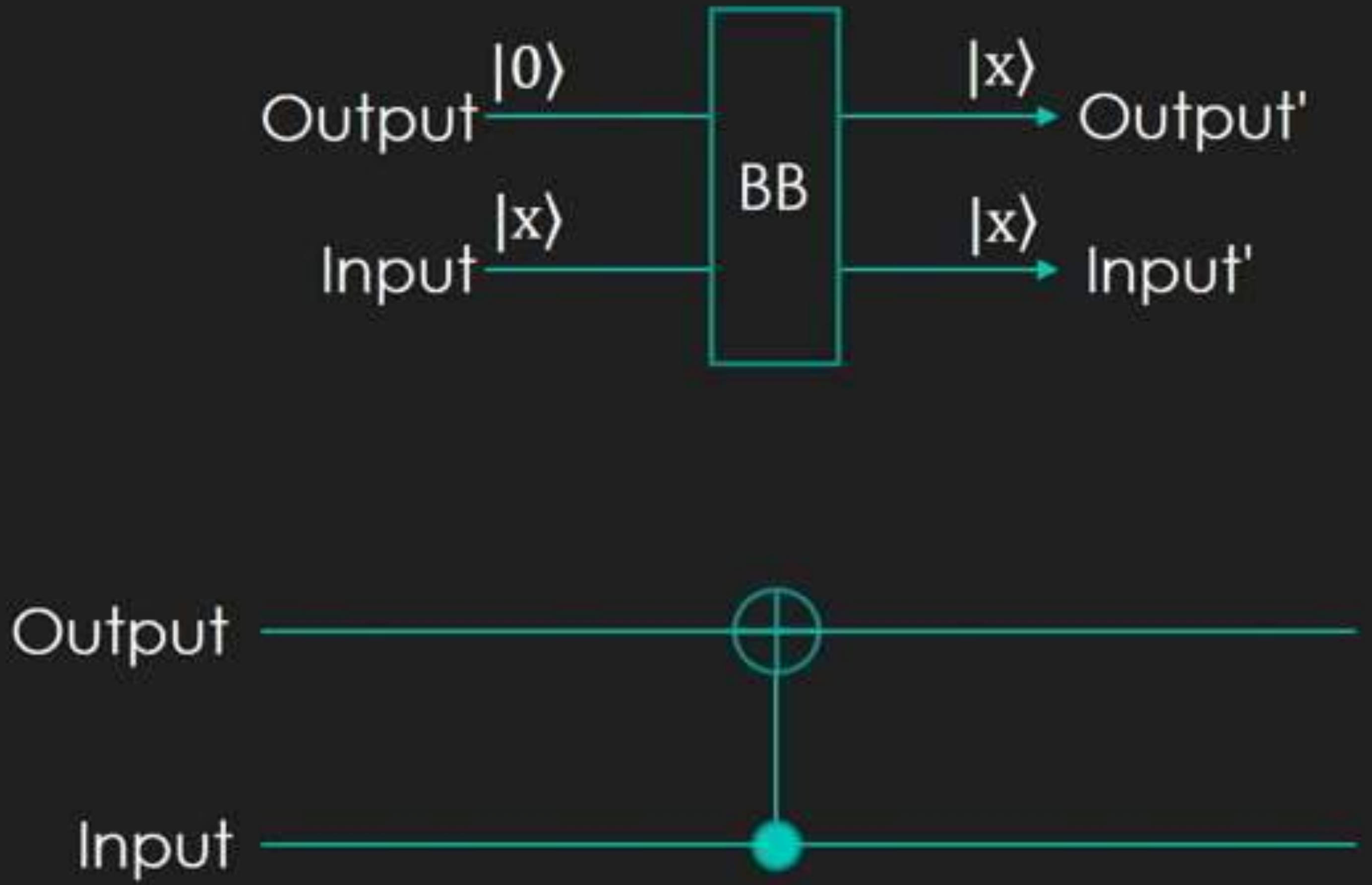
The Deutsch oracle: constant-1



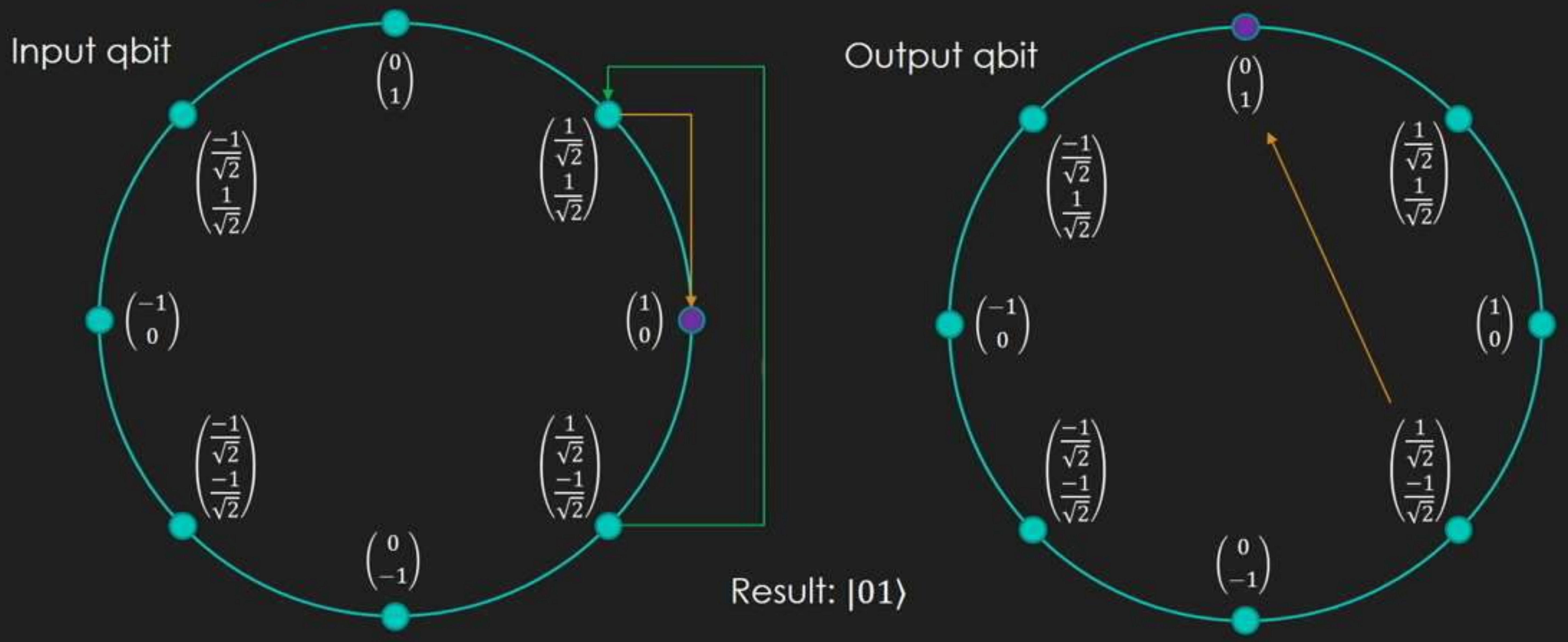
The Deutsch oracle: constant-1



The Deutsch oracle: identity



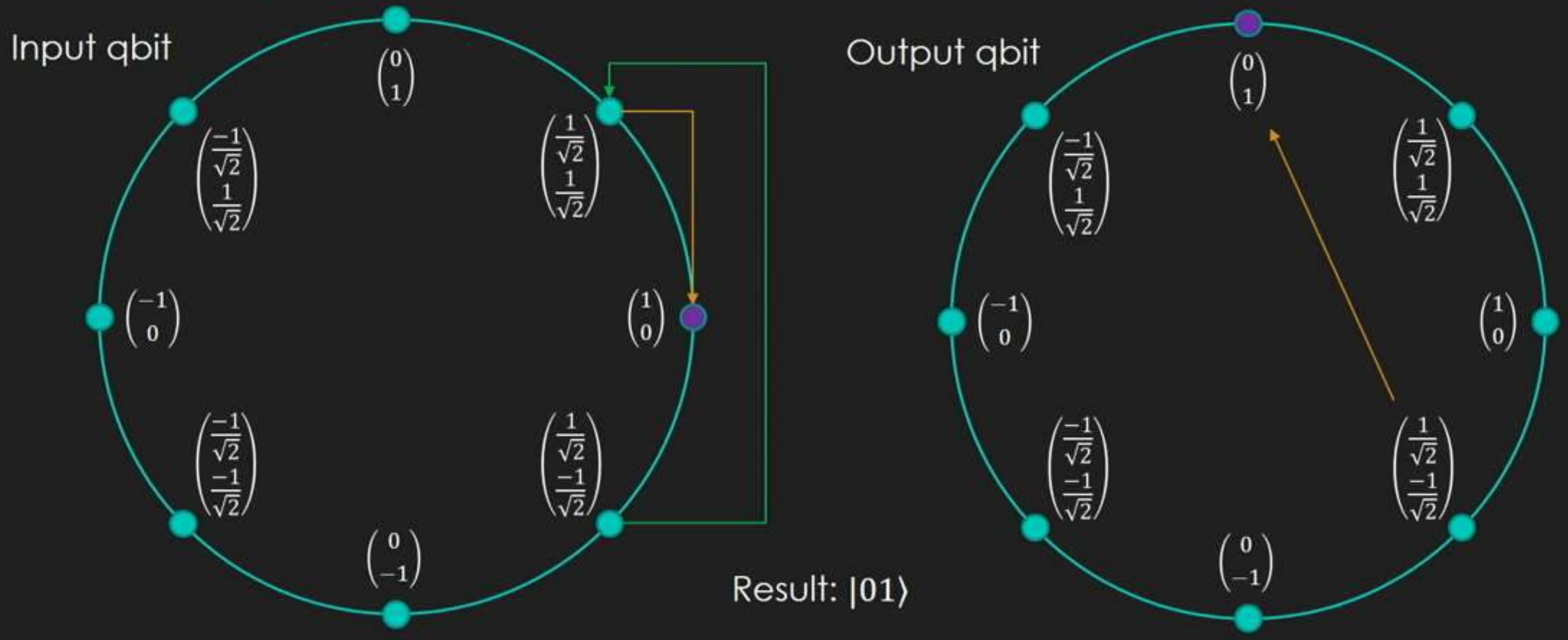
The Deutsch oracle: identity



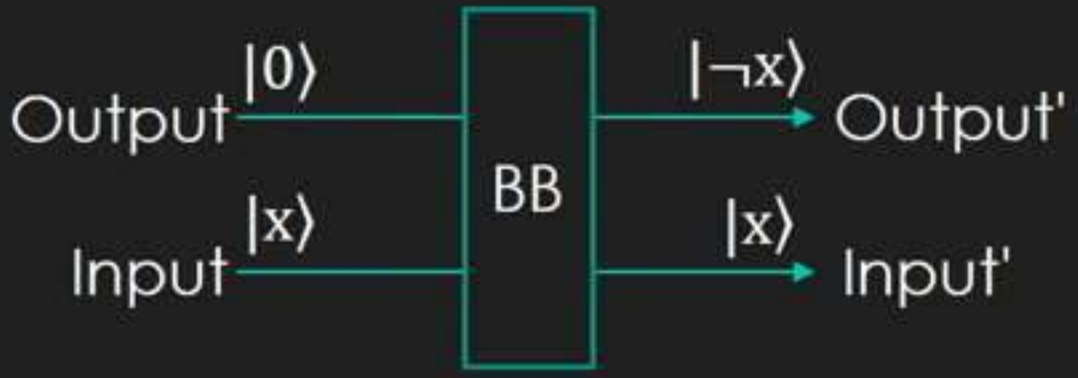
The Deutsch oracle: identity

$$C\left(\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\right)=C\begin{pmatrix}\frac{1}{2}\\ \frac{-1}{2}\\ \frac{-1}{2}\\ \frac{1}{2}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}$$

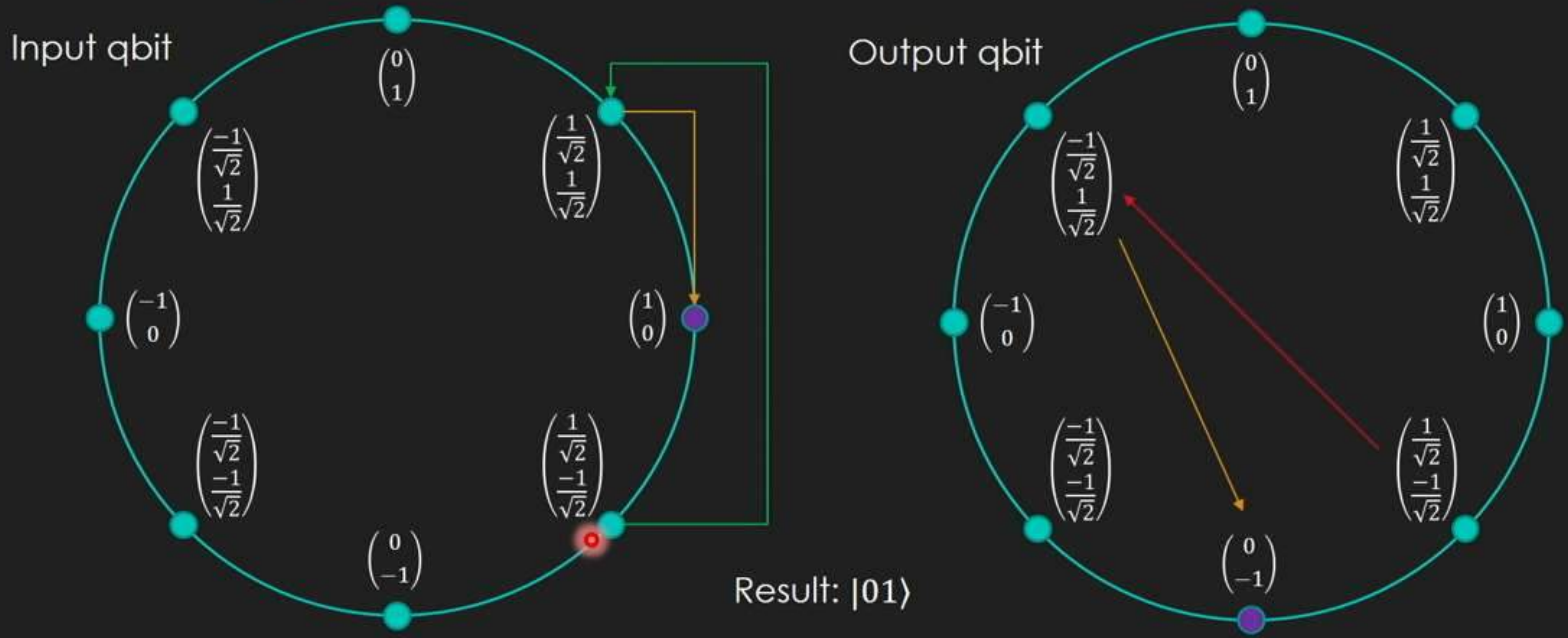
The Deutsch oracle: identity



The Deutsch oracle: negation



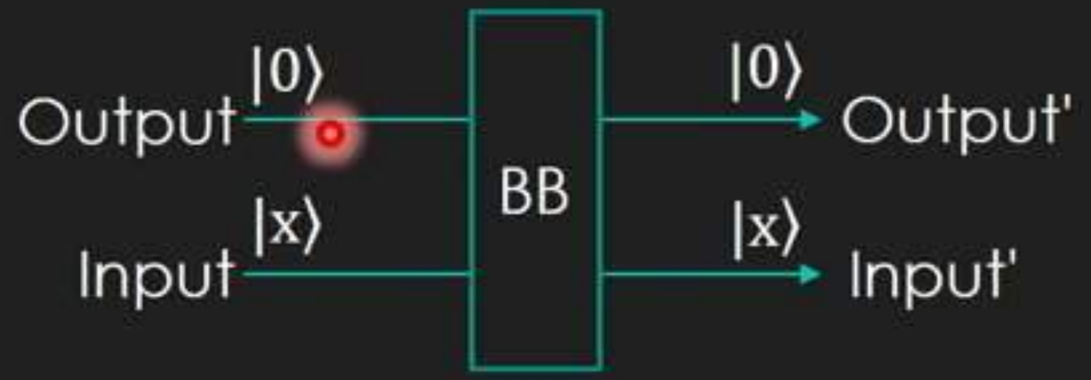
The Deutsch oracle: negation

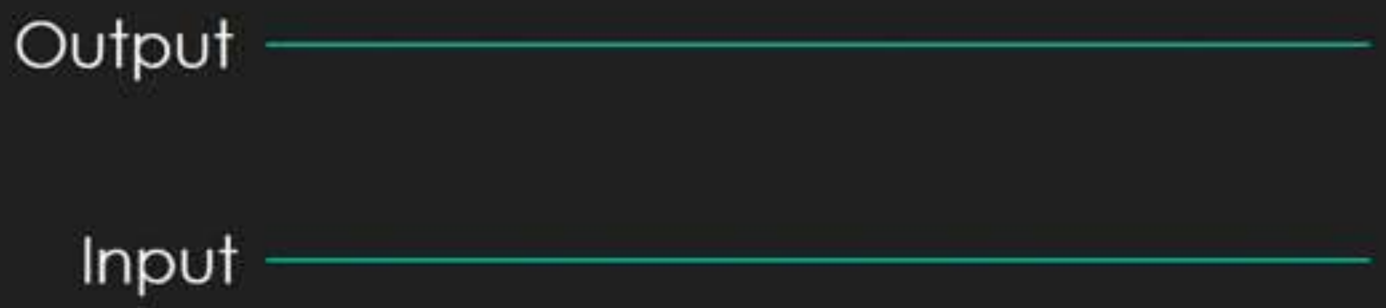


The Deutsch oracle

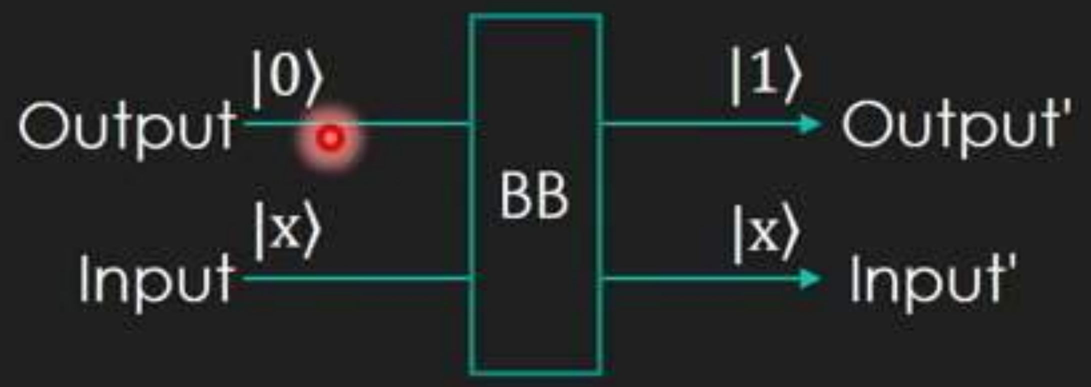
- We did it! We determined whether the function was constant or variable in a single query!
- Intuition: the difference *within* the categories (negation) was neutralized, while the difference *between* the categories (CNOT) was magnified
- This problem seems pretty contrived (and it was, when it was published)
- A generalized version with an n-bit black box also exists (Deutsch-Josza problem)
  - Determine whether the function returns the same value for all 2^n inputs (i.e. is constant)
- A variant of the generalized version was an inspiration for Shor's algorithm!

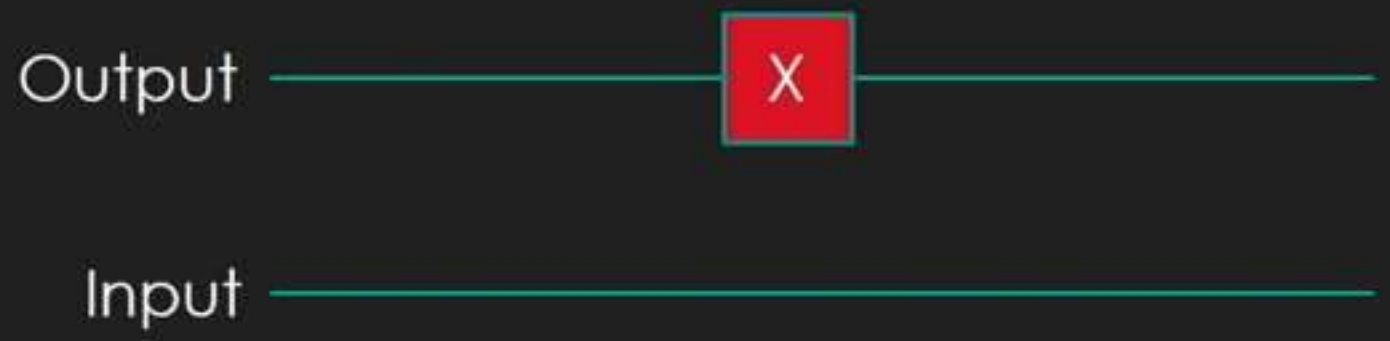
The Deutsch oracle: constant-0



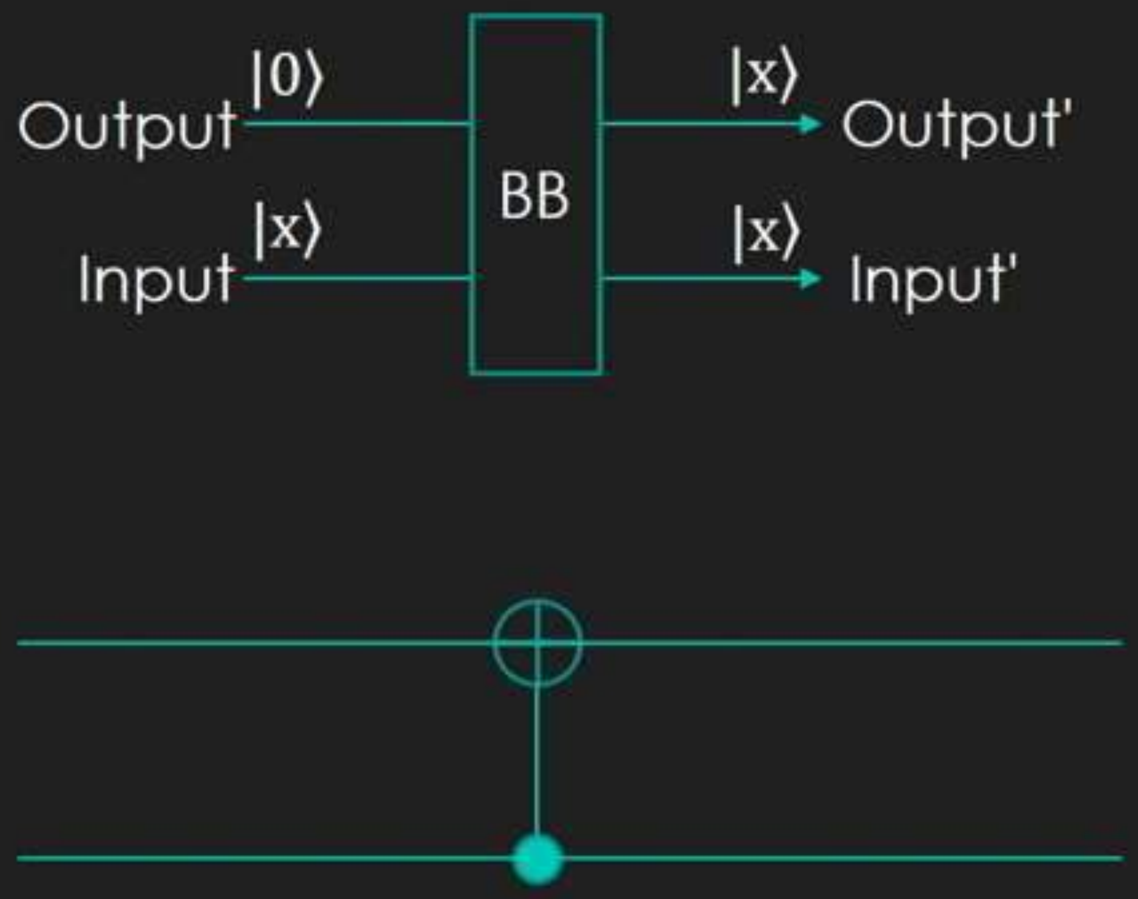


The Deutsch oracle: constant-1





The Deutsch oracle: identity



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Full recap

- We learned how to model classical computation with basic linear algebra
- We learned about qbits, superposition, and the Hadamard gate
- We learned the Deutsch Oracle problem, where quantum outperforms classical

Bonus topics

- Quantum entanglement
- Quantum teleportation

Entanglement

- If the product state of two qbits cannot be factored, they are said to be **entangled**

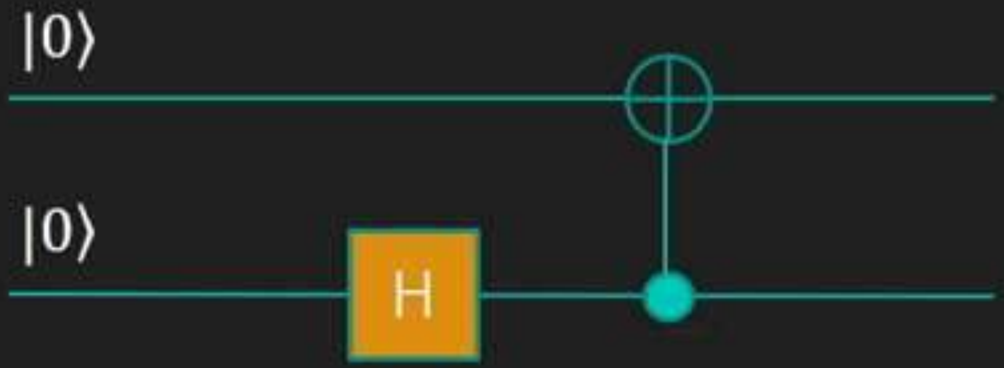
$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix}$$

$$ac = \frac{1}{\sqrt{2}}$$
$$ad = 0$$
$$bc = 0$$
$$bd = \frac{1}{\sqrt{2}}$$

- The system of equations has no solution, so we cannot factor the quantum state!
- This has a 50% chance of collapsing to $|00\rangle$ and 50% chance of collapsing to $|11\rangle$

Entanglement

How can we reach an entangled state? Easy!



$$CH_1\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right)=C\left(\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}$$

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$$\begin{aligned} ac &= \frac{1}{\sqrt{2}} \\ ad &= 0 \\ bc &= 0 \\ bd &= \frac{1}{\sqrt{2}} \end{aligned}$$

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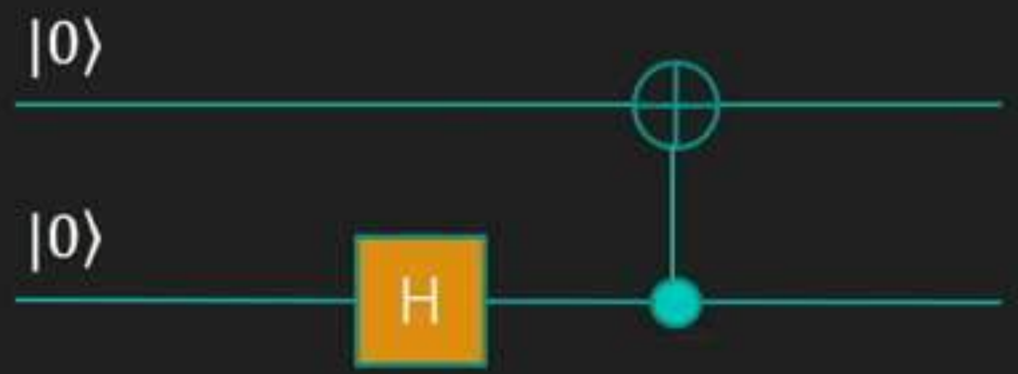

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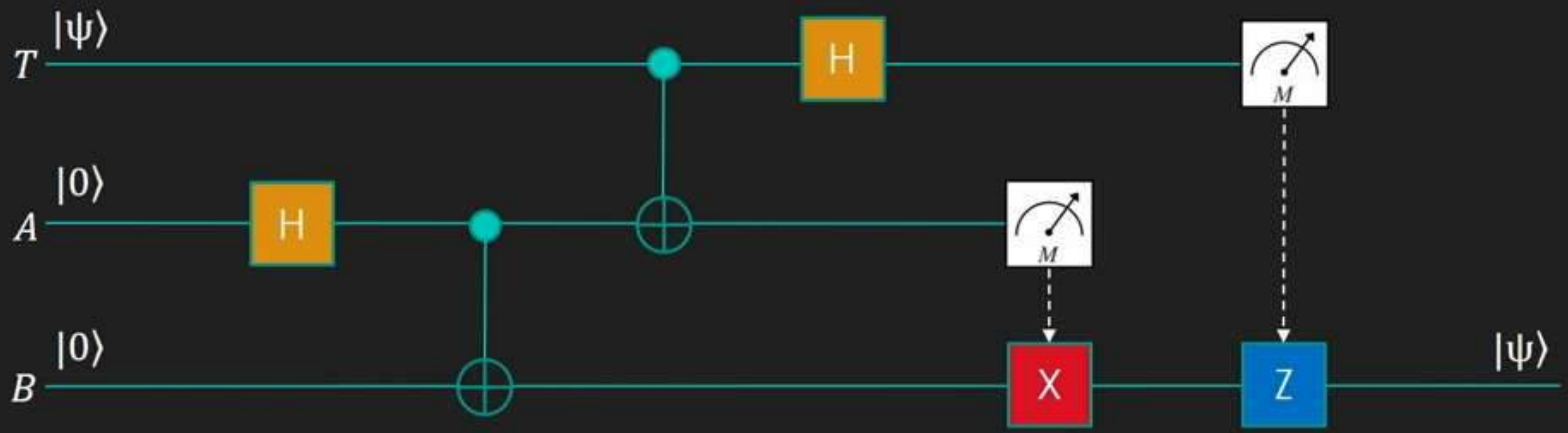
Entanglement

- What's going on here? The qbits seem to be coordinating in some way
  - Measuring one qbit also collapses the other in a correlated state
- This coordination happens even across vast stretches of space
- The coordination even happens faster than the speed of light! It is instantaneous.
  - A 2013 experiment measured particles within 0.01% of the travel time of light between them
- Surely the qbits "decided" at the time of entanglement what they would do?
  - No! This is called "hidden variable" theory and was disproved by John Bell in 1964
- This does indeed break locality through faster-than-light coordination
  - However – and this is the critical part – *no information can be communicated*

Teleportation

- **Quantum teleportation** is the process by which the state of an arbitrary qbit is transferred from one location to another by way of two other entangled qbits
- You can transfer qbit states (cut & paste) but you cannot clone them (copy & paste)
  - This is called the **No-cloning theorem**
- The teleportation is not faster-than-light, because some classical information must be sent

Teleportation



$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Further learning goals

- Deutsch-Jozsa algorithm and Simon's periodicity problem
  - Former yields oracle separation between EQP and P, latter between BQP and BPP
- Shor's algorithm and Grover's algorithm
- Quantum cryptographic key exchange
- How qbits, gates, and measurement are actually implemented
- Quantum error correction
- Quantum programming language design

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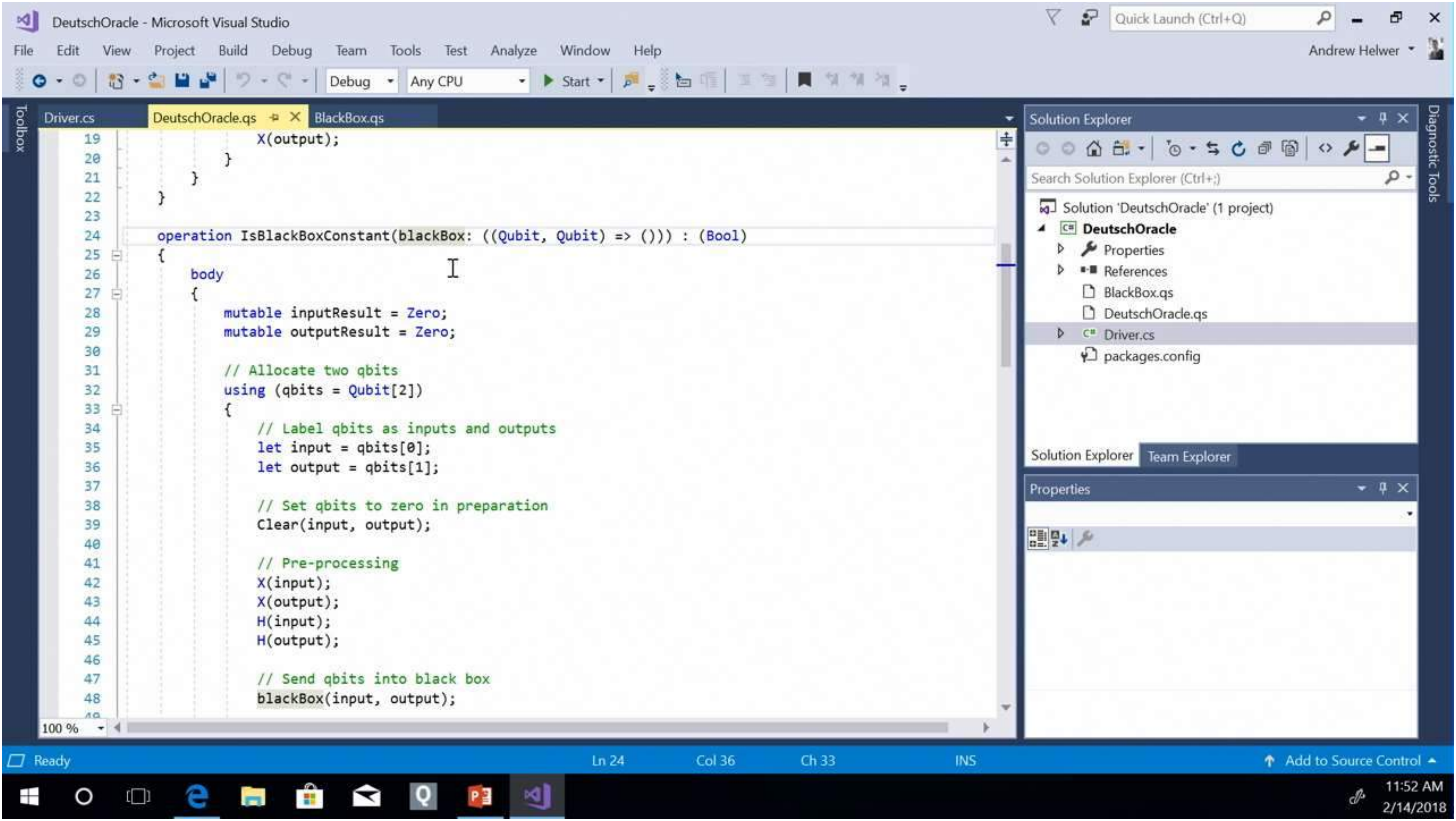
Further reading

- Recommended textbook: *Quantum Computing for Computer Scientists*
  - Others have recommended *Quantum Computing: A Gentle Introduction*
  - For those with heavier math backgrounds, *Quantum Computer Science: An Introduction*
- The Microsoft Quantum Development Kit docs are nice [link]
  - The development kit contains a quantum computer simulator!
  - Exercise: implement the Deutsch Oracle tester in Q#
- Some skepticism about physically-realizable quantum computers [link]
  - Noise might increase exponentially with the number of physical qbits

Appendices

- Single-bit operations on multi-bit states
- Quantum teleportation math


DeutschOracle - Microsoft Visual Studio
Quick Launch (Ctrl+Q)
File Edit View Project Build Debug Team Tools Test Analyze Window Help
Andrew Helwer
Debug
Any CPU
Start
Toolbox
Driver.cs
DeutschOracle.qs
BlackBox.qs
X(output);
}
}
}
operation IsBlackBoxConstant(blackBox: ((Qubit, Qubit) => ())) : (Bool)
{
body
{
mutable inputResult = Zero;
mutable outputResult = Zero;
// Allocate two qbits
using (qbits = Qubit[2])
{
// Label qbits as inputs and outputs
let input = qbits[0];
let output = qbits[1];
// Set qbits to zero in preparation
Clear(input, output);
// Pre-processing
X(input);
X(output);
H(input);
H(output);
// Send qbits into black box
blackBox(input, output);
100 %
Solution Explorer
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References
BlackBox.qs
DeutschOracle.qs
Driver.cs
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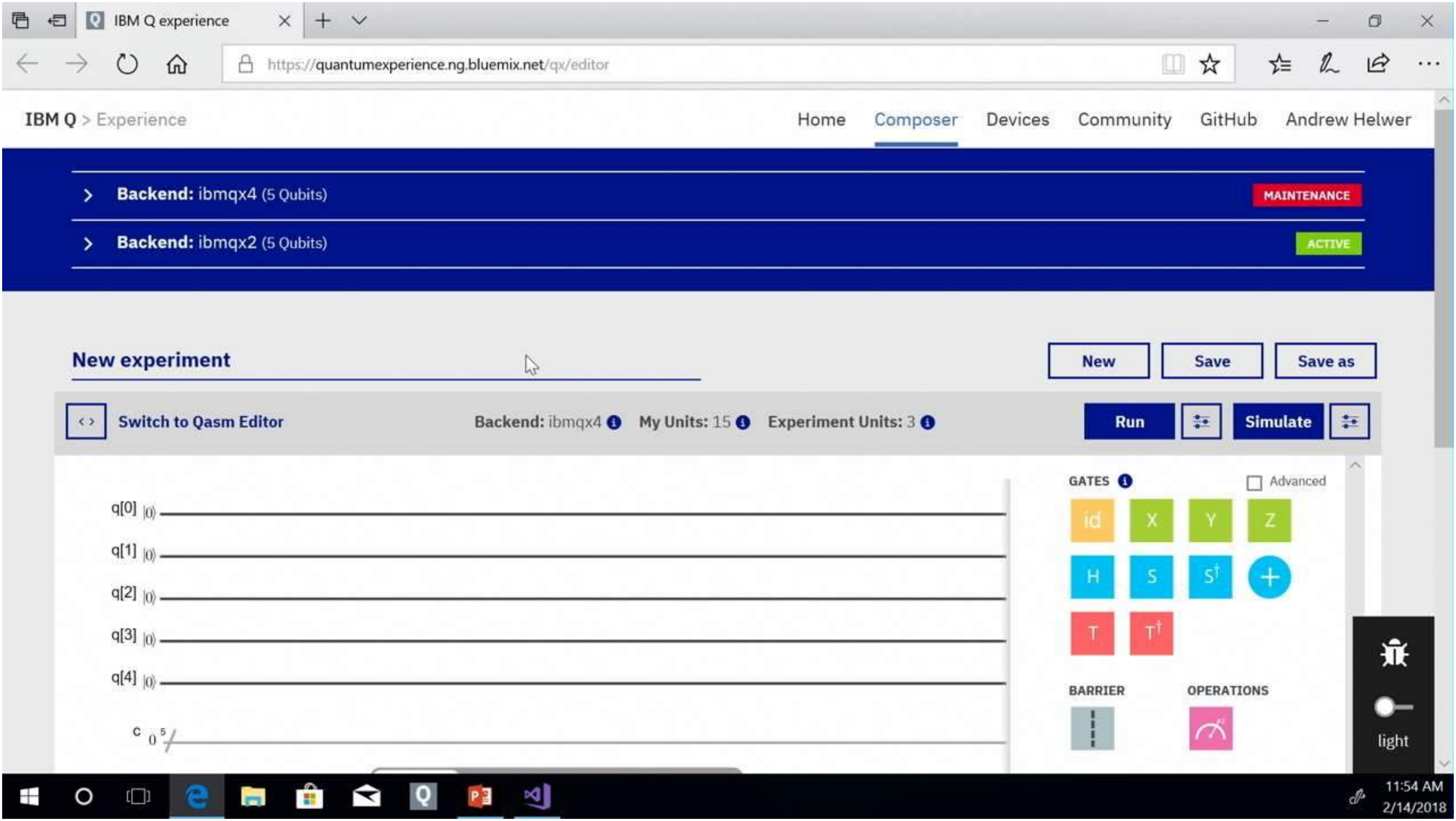
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IBM Q > Experience
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Composer
Devices
Community
GitHub
Andrew Helwer
Backend: ibmqx4 (5 Qubits)
MAINTENANCE
Backend: ibmqx2 (5 Qubits)
ACTIVE
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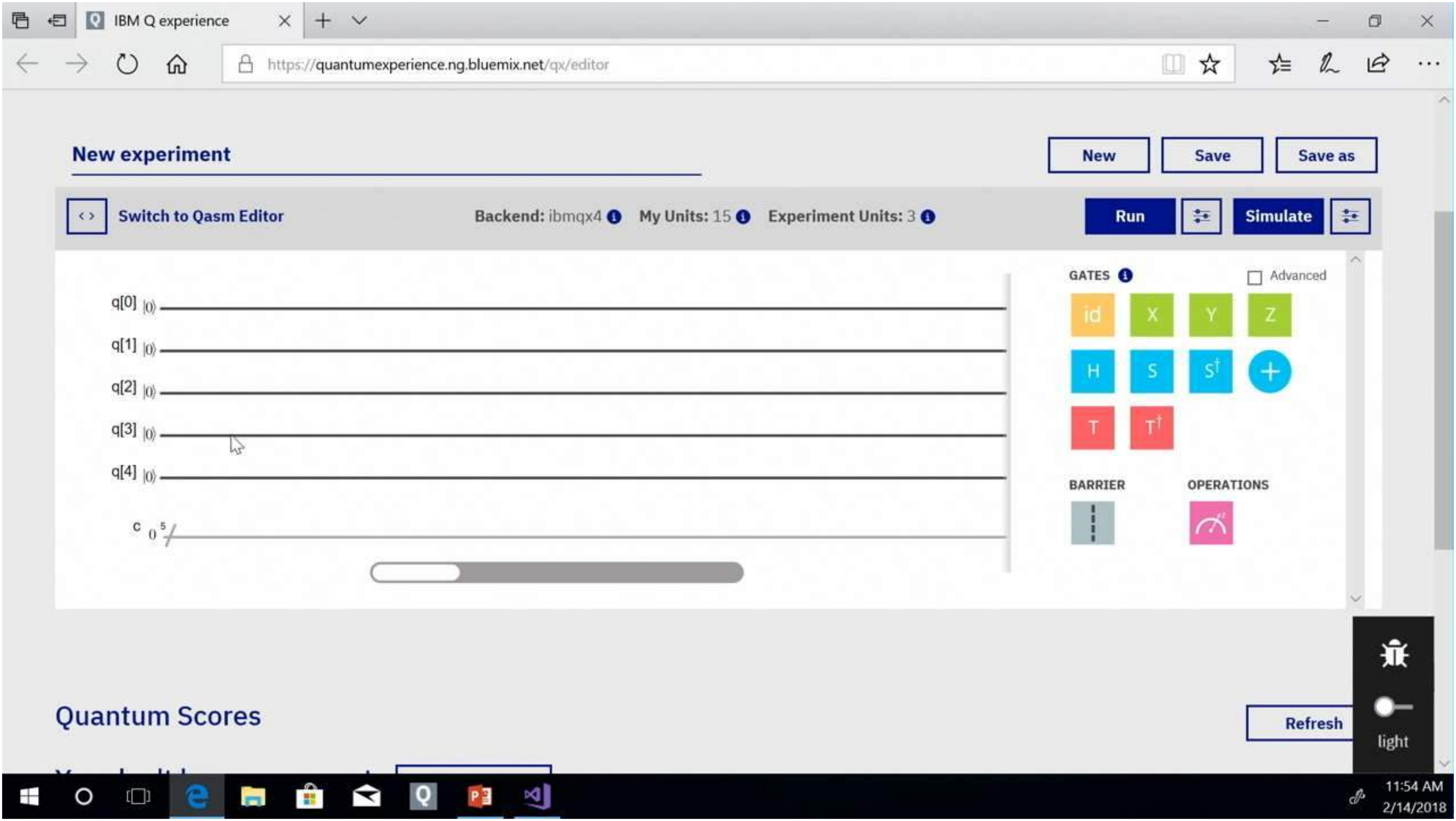
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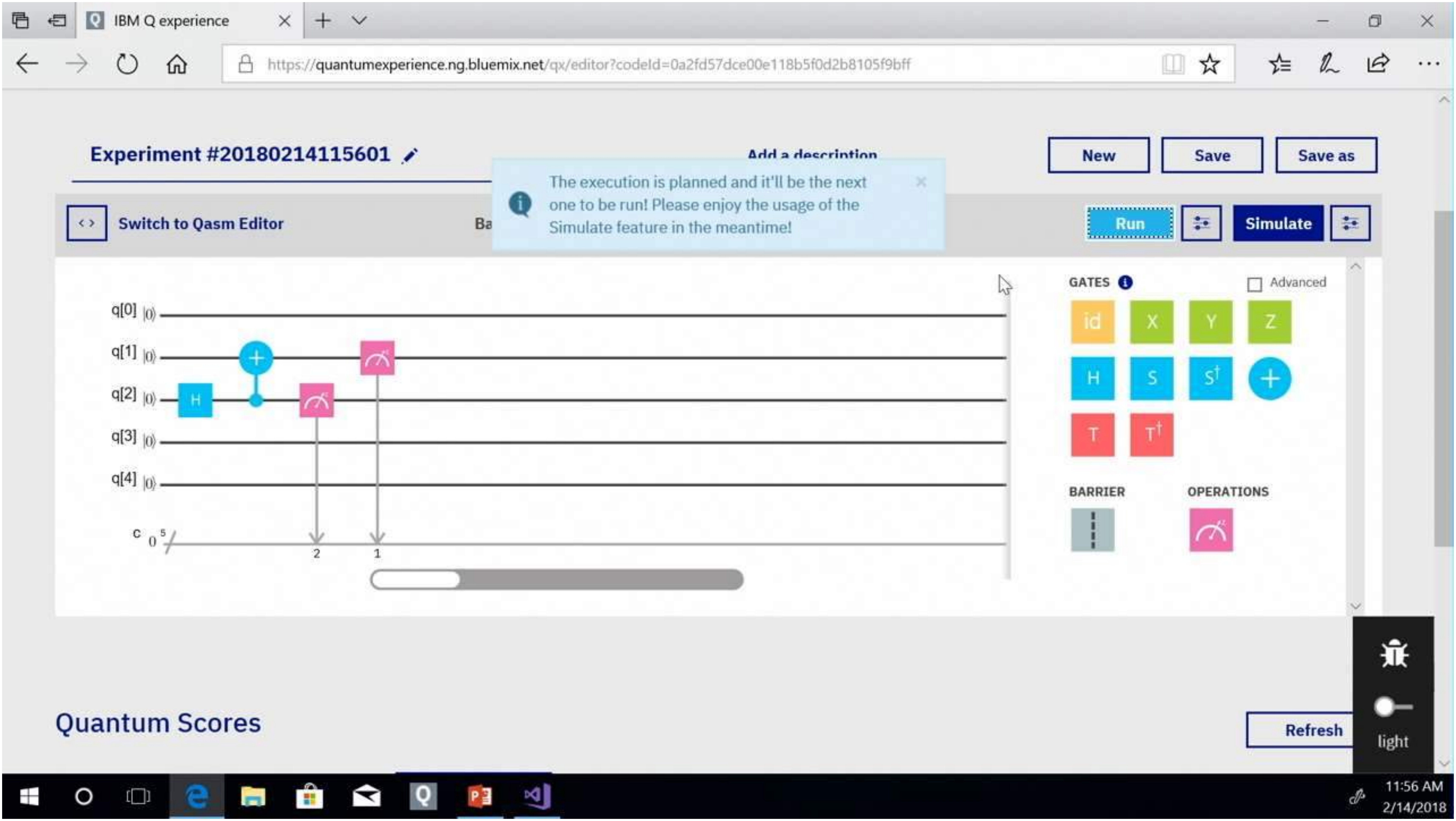
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The execution is planned and it'll be the next one to be run! Please enjoy the usage of the Simulate feature in the meantime!
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q[2] |0⟩
q[3] |0⟩
q[4] |0⟩
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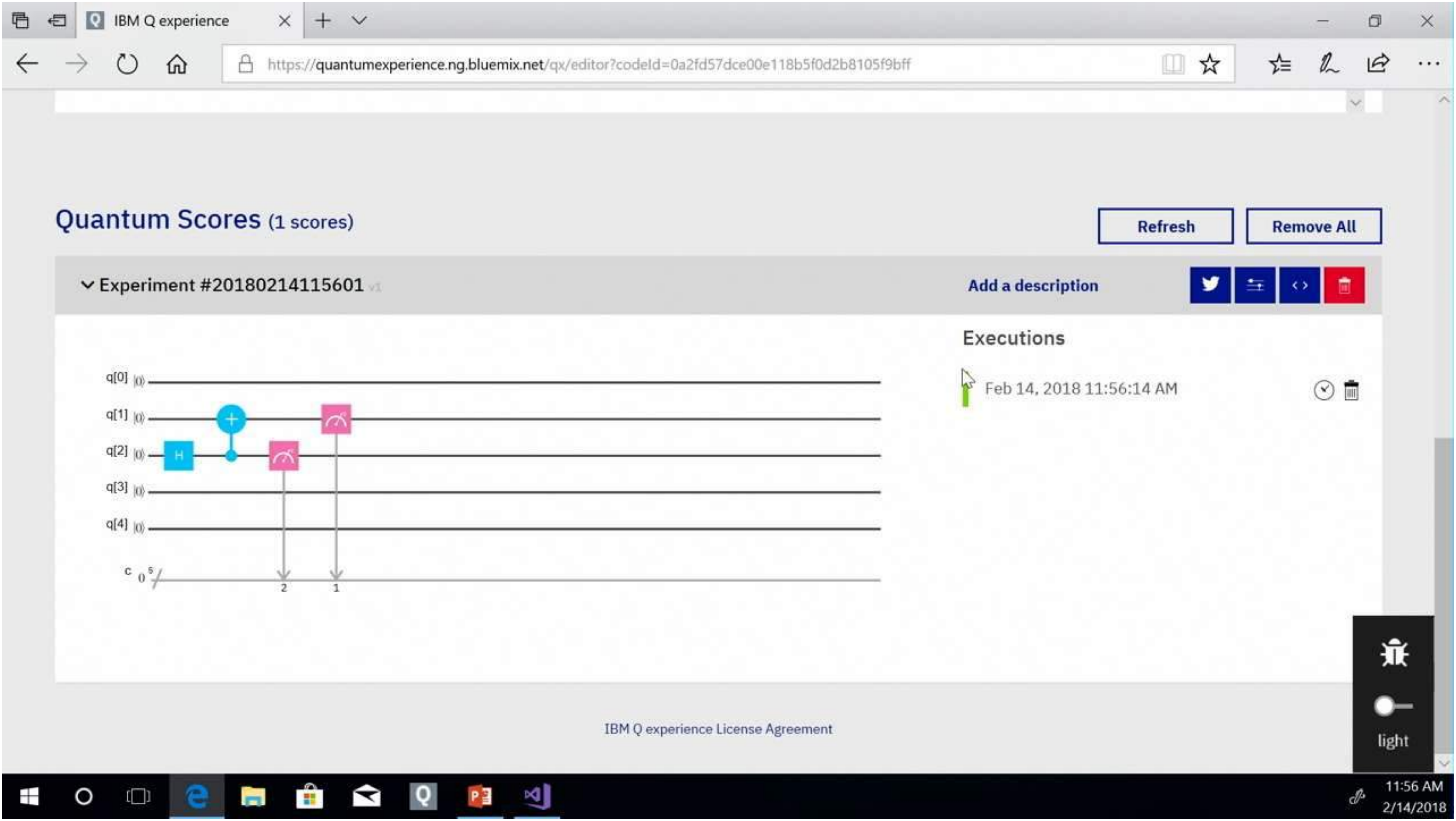
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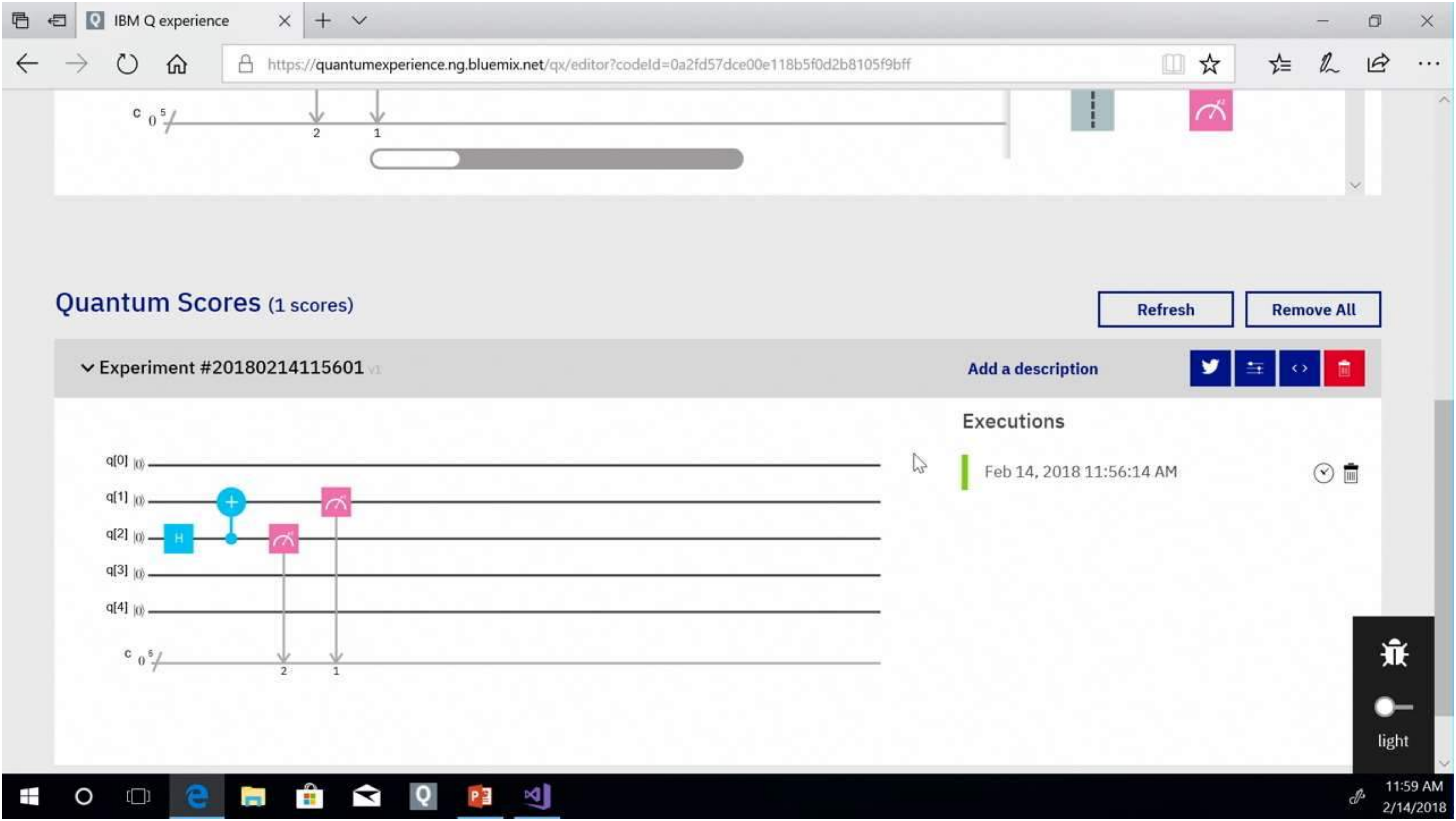
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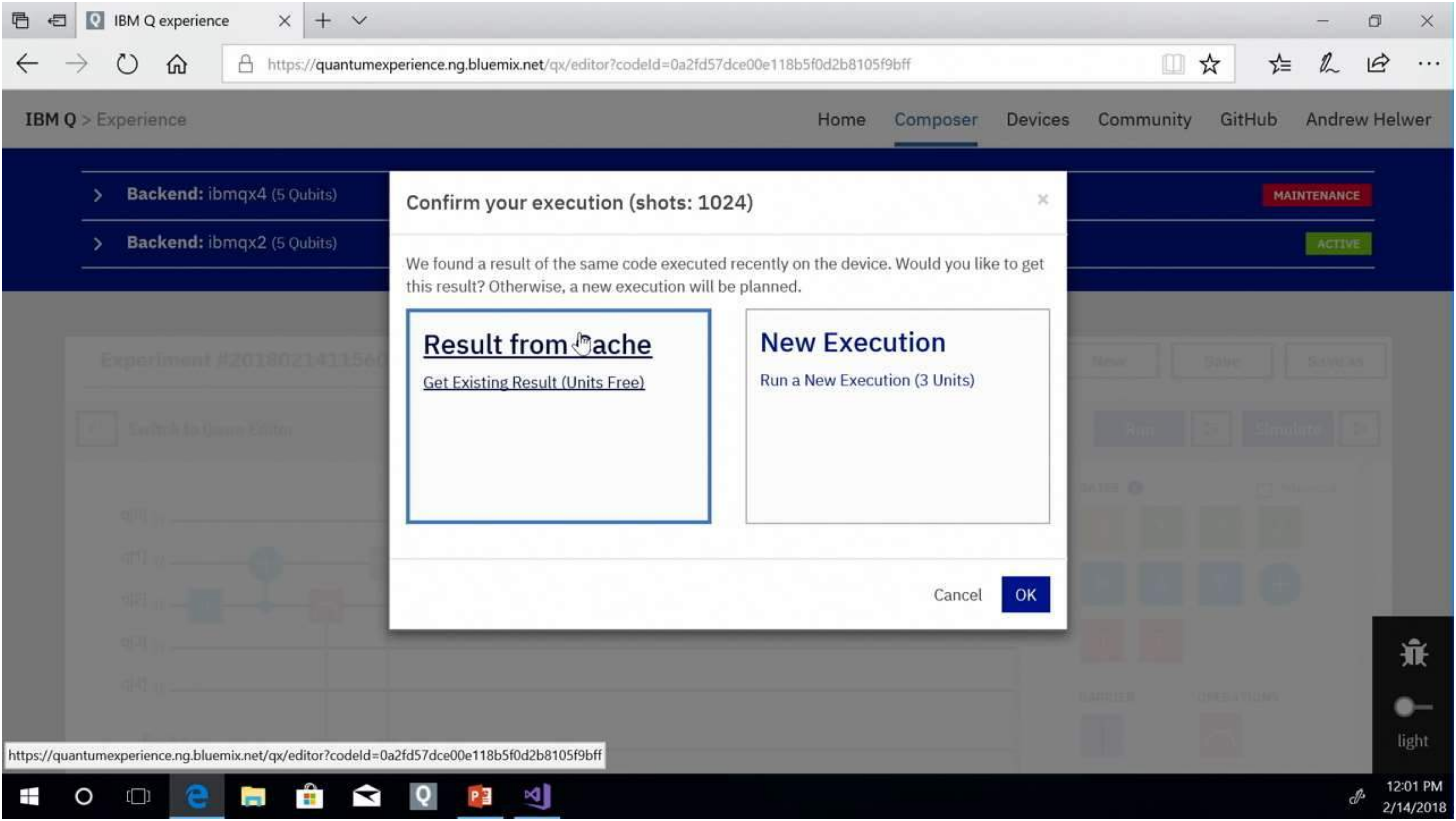
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Andrew Helwer
Backend: ibmqx4 (5 Qubits)
MAINTENANCE
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ACTIVE
Confirm your execution (shots: 1024)
We found a result of the same code executed recently on the device. Would you like to get this result? Otherwise, a new execution will be planned.
Result from Cache
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OK
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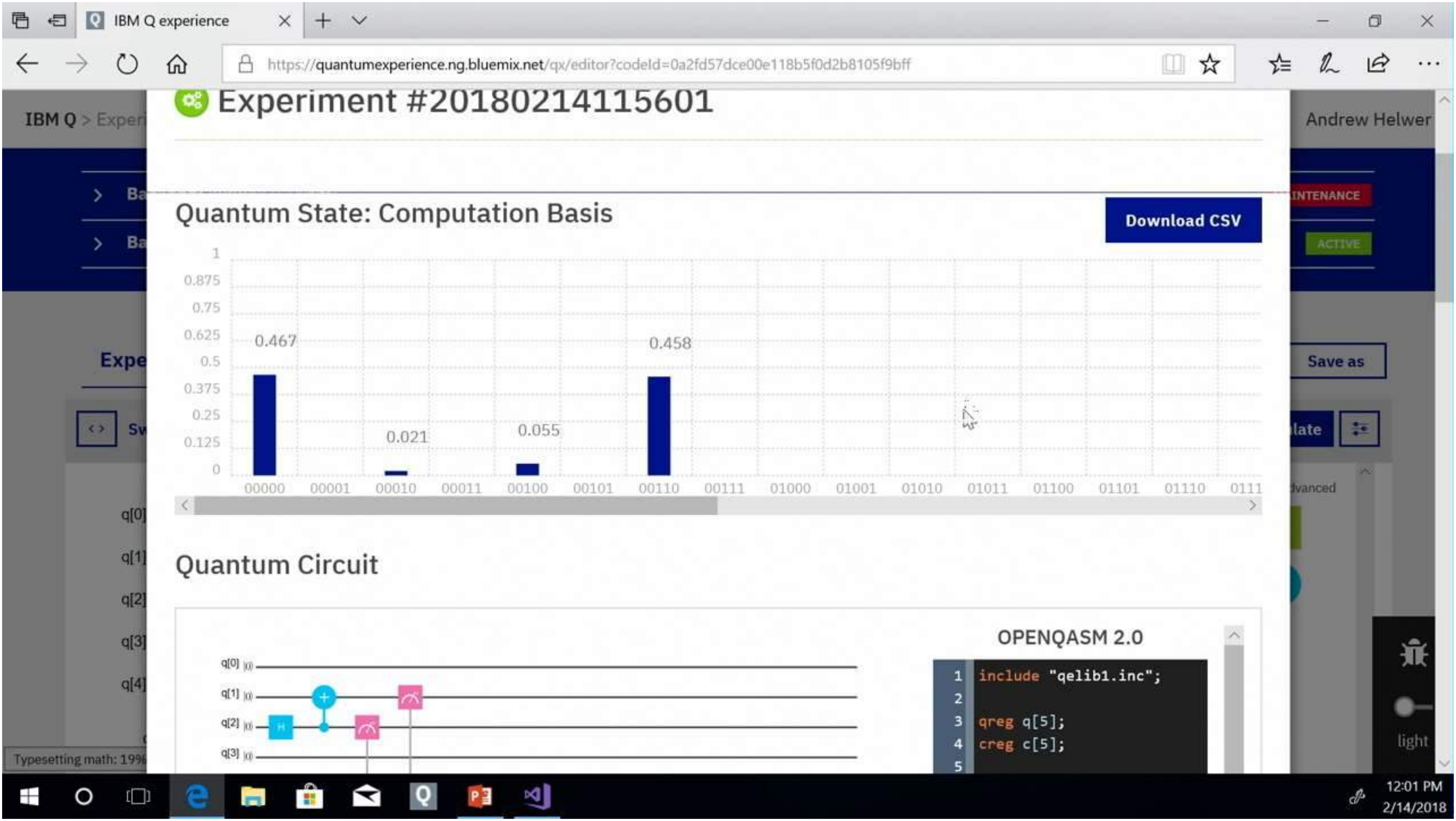
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Experiment #20180214115601
Quantum State: Computation Basis
Download CSV
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01011
01100
01101
01110
0111
Quantum Circuit
q[0] |0⟩
q[1] |0⟩
q[2] |0⟩
q[3] |0⟩
H
OPENQASM 2.0
1 include "qelib1.inc";
2
3 qreg q[5];
4 creg c[5];
5
IBM Q
Andrew Helwer
Save as
Typesetting math: 19%
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2/14/2018